

# Casimir interaction of rod-like particles in a two-dimensional critical system

E. Eisenriegler<sup>1</sup> and T. W. Burkhardt<sup>2</sup>

<sup>1</sup>*Theoretical Soft Matter and Biophysics, Institute of Complex Systems,  
Forschungszentrum Jülich, D-52425 Jülich, Germany*

<sup>2</sup>*Department of Physics, Temple University, Philadelphia, PA 19122, USA*

(Dated: February 16, 2016)

## Abstract

We consider the fluctuation-induced interaction of two thin, rod-like particles or “needles” immersed in a two-dimensional critical fluid of Ising symmetry right at the critical point. Conformally mapping the plane containing the needles onto a simpler geometry in which the stress tensor is known, we analyze the force and torque between needles of arbitrary length, separation, and orientation. For infinite and semi-infinite needles we utilize the mapping of the plane bounded by the needles onto the half plane, and for two needles of finite length the mapping onto an annulus. For semi-infinite and infinite needles the force is expressed in terms of elementary functions, and we also obtain analytical results for the force and torque between needles of finite length with separation much greater than their length. Evaluating formulas in our approach numerically for several needle geometries and surface universality classes, we study the full crossover from small to large values of the separation to length ratio. In these two limits the numerical results agree with results for infinitely long needles and with predictions of the small-particle operator expansion, respectively.

## I. INTRODUCTION

Two objects immersed in a near-critical fluid, for example colloidal particles in a binary liquid mixture near the critical point of miscibility, experience a long-range, fluctuation-induced force [1–6]. Changes in the positions of the objects alter the space available to the critically fluctuating fluid, and hence its free energy, giving rise to an effective interaction of the objects. In analogy with the Casimir effect in quantum electrodynamics [7–10], this is known as the critical or thermodynamic Casimir interaction.

The critical Casimir interaction displays a high degree of universality, i.e. is largely independent of microscopic details [3–6]. It only depends on universal properties of the fluid, the universality class of the boundary between the fluid and the immersed particles, and geometrical properties of the particles, such as their size, shape, and relative position.

Particles immersed in a solution [11–14] of long, flexible polymer chains or particles to which a polymer chain is attached [14, 15] experience a similar Casimir interaction due to fluctuations of the polymers [16]. Fluctuations of the superfluid order parameter also lead to critical Casimir forces, and this has been detected in wetting films of  $^4\text{He}$  [17–19].

Binary liquid mixtures belong to the Ising universality class. The surfaces of the immersed particles generally attract one of the two components of the mixture preferentially, corresponding to (+ or  $-$ ) boundary conditions in the Ising model. A surface prepared to suppress the preference corresponds to free-spin boundary conditions [20]. In the terminology of surface critical phenomena these two surface universality classes [21] are known as “normal” (+ or  $-$ ) and “ordinary” ( $O$ ).

In studies of critical Casimir interactions, systems with planar walls and systems with spherical particles have received the most attention [3, 19, 22–24]. For non-spherical particles the Casimir interaction depends on their orientation as well as their separation, i.e., there is a torque as well as a force. Recently the universal scaling form of the Casimir interaction of a prolate uniaxial ellipsoid and a planar wall, with pairs ++ or +- of boundary conditions on the two surfaces, was calculated within mean-field theory by Kondrat et al. [25].

In this paper we derive exact results for the Casimir interaction of two rod-like particles in a two-dimensional critical system in the Ising universality class. The following considerations provide some motivation:

- (i) Recent experiments suggest that biological membranes are tuned close to a critical

point of miscibility in two dimensions [26]. The possibility of critical Casimir interactions between inclusions in the membrane has been studied by Machta et al. [27].

(ii) Systems at the critical point are generally invariant not only under scale transformations, but also under conformal or angle-preserving coordinate transformations [28]. This has far-reaching consequences for the Casimir interaction of two particles, especially in two spatial dimensions [24, 29, 30].

In general dimension  $d$  the region outside two spherical particles with arbitrary radii and separation can be conformally mapped onto the region bounded by two concentric spheres using homogeneous translations, rotations, and dilatations and the inversion. Burkhardt, Eisenriegler, and Ritschel have shown [24] that this mapping determines the asymptotic form of the Casimir interaction both for large and small separation of the spheres in an arbitrary critical medium, not necessarily Ising-like, in arbitrary spatial dimension  $d$ .

In  $d = 2$  the conformal group is much richer than in general  $d$ . Conformal mappings are generated by analytic functions, and the doubly connected region outside two particles of *arbitrary* shape can be conformally mapped onto the annulus bounded by two concentric circles or, equivalently, onto the surface of a cylinder of finite circumference and length. The Casimir interaction of the particles in an infinite, two-dimensional, Ising-like critical medium follows from the free energy of the critical Ising model on the cylindrical surface [24, 28–30], which Cardy [31] has derived in analytic form for all aspect ratios and for all pairs of boundary conditions  $+$ ,  $-$ , and  $O$  at the ends of the cylindrical surface. Burkhardt and Eisenriegler [24] and Machta et al. [27] followed this route in evaluating the Casimir interaction of two particles with circular shape. Bimonte et al. [30] have given a general analysis of asymptotic properties of the Casimir interaction in critical two-dimensional conformal field theories for two particles of arbitrary shape, based on the mapping of the portion of the plane outside the particles onto an annulus.

Being interested in the Casimir torque as well as the force, we consider anisotropic particles. The rod-like particles in our study have negligible width compared with their length, and we model each particle as a segment of a straight line. This shape is extremely simple and highly-anisotropic. Following Ref. [29] we refer to the particles as “needles” [32]. The approach we use, which makes use of conformal mappings, is not limited to these needles but is applicable, in principle, to particles in two dimensions of arbitrary shape.

In Section II and Appendix A we show how the force and torque between the particles

are related to the stress tensor [28], which for our purpose is the quantity most convenient to work with [24, 29, 30].

In Section III we calculate the critical Casimir force between a semi-infinite needle and an infinite needle and between two semi-infinite needles, with arbitrary relative position. These are instructive cases to begin with since the calculations can be carried out analytically, without special functions. The key step is to generate by a conformal mapping  $z(w)$  the complex  $z$  plane with the two embedded needles from the upper half  $w$  plane with the corresponding needles on the real axis.

In Section IV we discuss the more complicated Schwarz-Christoffel transformation required for two needles of finite length. It generates the complex  $z$  plane, with an embedded needle between points  $z_1$  and  $z_2$  and a second needle between points  $z_3$  and  $z_4$ , from an annulus, with circular needles on the outer and inner boundaries.

In Section V detailed results for the Casimir force and torque between needles of finite length are presented for several configurations of the needles. The results are consistent with predictions of Vasilyev et al. [29], who have studied the Casimir interaction of the needles with Monte Carlo simulations and calculated them in certain symmetric cases with conformal invariance methods, but without the generality of the approach considered below. The results of Section V are also asymptotically consistent with the predictions for infinite or semi-infinite needles derived in Section III. Since the torque diverges for needles of infinite length, checking its asymptotic behavior is more subtle and is addressed in Appendix C.

For large separation of the needles in comparison with their lengths [33], the numerical results of Section V reproduce the predictions of the “small-particle operator expansion” (SPOE). This expansion, which is reviewed in Appendix B, has proved to be extremely useful in studies of the critical Casimir interaction and is similar in spirit to the operator product expansion [34] in field theory. Large needle separation corresponds to a small ratio of inner to outer radius of the annulus, and the corresponding expansions in Section IV and Appendix B allow us to check the agreement with the SPOE analytically.

The final section of the paper, Section VI, contains a summary and concluding remarks.

## II. FORCE, TORQUE, AND THE STRESS TENSOR

In this section the force and torque acting on needles in the  $z$  plane are expressed in terms of the average complex stress tensor  $\langle T(z) \rangle$ . In subsequent sections these three quantities are calculated for several two-needle geometries of interest. In each case  $\langle T(z) \rangle$  is obtained through a conformal mapping  $z(w)$  of a simpler geometry in the  $w$  plane, for which  $\langle T(w) \rangle$  is known, onto the desired geometry in the  $z$  plane, using the fundamental transformation property [28]

$$\langle T(z) \rangle = \frac{1}{(z'(w))^2} \left[ \langle T(w) \rangle - \frac{1}{24} S(w) \right]. \quad (2.1)$$

Here  $S(w)$  is the Schwarzian derivative

$$S(w) = \frac{z'''(w)z'(w) - (3/2)(z''(w))^2}{(z'(w))^2} \equiv \frac{d^2}{dw^2} \ln \frac{dz}{dw} - \frac{1}{2} \left( \frac{d}{dw} \ln \frac{dz}{dw} \right)^2, \quad (2.2)$$

and the primes denote derivatives.

The transformation of the free energy under an arbitrary infinitesimal coordinate transformation is reviewed in Appendix A. According to Eq. (A11) the components  $f_x$  and  $f_y$  of the force on needle  $I$  due to a second needle  $II$  are given by

$$(f_x, f_y)/(k_B T) = -(\text{Im}, \text{Re})\tau, \quad (2.3)$$

where

$$\tau = \frac{1}{\pi} \int_{\mathcal{C}_I} dz \langle T(z) \rangle = \frac{1}{\pi} \int_{\mathcal{C}} dw z'(w) \langle T(w) \rangle. \quad (2.4)$$

The integration path  $\mathcal{C}_I$  in the  $z$  plane encloses needle  $I$  in a clockwise fashion, leaving needle  $II$  outside, and  $\mathcal{C}$  is the corresponding path in the  $w$  plane, which maps onto  $\mathcal{C}_I$  under the conformal transformation  $z(w)$ . With the help of Eq. (2.1),  $\tau$  can be expressed as

$$\tau = \frac{1}{\pi} \int_{\mathcal{C}} dw \frac{1}{z'(w)} \left[ \langle T(w) \rangle - \frac{1}{24} S(w) \right] \equiv \tau^{(T)} + \tau^{(S)}. \quad (2.5)$$

We define the torque  $\Theta$  on a needle  $I$  with fixed length, extending from  $z_1$  to  $z_2$ , and forming an angle  $\Phi_I \equiv \Phi_{12} = \arg(z_1 - z_2)$  with the  $x$  axis, by

$$\Theta = -(\partial/\partial\Phi_I)\delta F. \quad (2.6)$$

Here  $\delta F$  is the free energy of interaction [35], and the derivative is taken for an infinitesimal rotation of needle  $I$  about its midpoint with the midpoint  $z_I = \frac{1}{2}(z_1 + z_2)$  fixed. According to Eqs. (2.6) and (A12), the torque may be written as

$$\Theta = -k_B T \operatorname{Re} \theta, \quad (2.7)$$

where

$$\theta = \frac{1}{\pi} \int_{\mathcal{C}_I} dz \langle T(z) \rangle (z - z_I) \quad (2.8)$$

and the integration path  $\mathcal{C}_I$  is the same as in Eq. (2.4). Since

$$z - z_n = \int_{w_n}^w d\tilde{w} (dz/d\tilde{w}) \equiv \zeta_n(w); \quad n = 1, 2, \quad (2.9)$$

where  $w_n$  is the point in the  $w$  plane which maps onto needle endpoint  $z_n \equiv z(w_n)$ , the torque may be expressed as

$$\Theta = -k_B T \operatorname{Re} \left( \theta^{(T)} + \theta^{(S)} \right), \quad (2.10)$$

where

$$\left\{ \theta^{(T)}, \theta^{(S)} \right\} = \frac{1}{2\pi} \int_{\mathcal{C}} dw \frac{1}{dz/dw} \left\{ \langle T(w) \rangle, -\frac{1}{24} S(w) \right\} \left[ \zeta_1(w) + \zeta_2(w) \right], \quad (2.11)$$

in terms of  $\langle T(w) \rangle$ . Note that the contributions  $\tau^{(S)}$  in Eq. (2.5) and  $\theta^{(S)}$  in Eq. (2.11) which involve the Schwarzian derivative are purely geometrical and do not depend on the surface universality classes of the needles.

### III. INTERACTIONS OF INFINITE AND SEMI-INFINITE NEEDLES

#### A. Force between a semi-infinite and an infinite needle

In this and subsequent sections we use the notation  $z = r_x + ir_y$  and  $w = \rho_u + i\rho_v$  for the complex variables  $z$  and  $w$  and their real and imaginary parts.

The conformal transformation  $z(w)$ , where

$$z'(w) = \mathcal{A}e^{i\alpha}w^{-\alpha/\pi-1}(w-1), \quad (3.1)$$

$$z(w) = \pi\mathcal{A}e^{i\alpha}w^{-\alpha/\pi}\left(\frac{w}{\pi-\alpha} + \frac{1}{\alpha}\right), \quad (3.2)$$

$\mathcal{A}$  is a positive real constant, and  $0 < \alpha < \pi$ , considered in §12.1 of Kober [36], maps the upper half  $w$  plane, with semi-infinite needles along the positive and negative  $u$  axes, onto the upper half  $z$  plane with two transformed needles. Needle *II*, the image of the negative  $u$  axis, is infinitely long and corresponds to the entire  $x$  axis. Needle *I*, the image of the positive  $u$  axis, is semi-infinite and extends from the point

$$z(1) = \frac{\pi^2}{\alpha(\pi-\alpha)}\mathcal{A}e^{i\alpha} \quad (3.3)$$

to  $\infty$ , forming an angle  $\alpha$  with the  $x$  axis.

The integrand in expressions (2.3), (2.4), and (2.5) for the Casimir force follows from Eqs. (2.2), (3.1), and the corresponding stress tensor [37]

$$\langle T(w) \rangle = \frac{\tilde{t}}{w^2} = \frac{(0, 0, \frac{1}{2}, \frac{1}{16})}{w^2} \quad (3.4)$$

for  $(OO, ++, +-, O+)$  boundary conditions on the two needles and is given by

$$\begin{aligned} z'(w) \langle T(z) \rangle &= -\frac{e^{-i\alpha}}{48\pi^2\mathcal{A}} \frac{w^{\alpha/\pi-1}}{(w-1)^3} \\ &\times \left[ \alpha(2\pi-\alpha)w^2 - 2(2\pi-\alpha)(\pi+\alpha)w + \pi^2 - \alpha^2 - 48\pi^2\tilde{t}(w-1)^2 \right]. \end{aligned} \quad (3.5)$$

In Eq. (2.4) integrating clockwise along the edges of needle *I* in the  $z$  plane (path  $\mathcal{C}_I$ ) corresponds to integrating along the  $u$  axis from  $\rho_u = 0$  to  $+\infty$  in the  $w$  plane (path  $\mathcal{C}$ ). Combining Eqs. (2.4) and (3.5), evaluating the integral, and making use of Eqs. (2.3) and (3.3), we obtain  $f_x = 0$ , as expected since needle *II* is infinite, and [38]

$$\tau = -\frac{f_y}{k_B T} = \frac{1}{96r_y(1)} \frac{(2\pi-\alpha)(\pi+\alpha) - 96\pi^2\tilde{t}}{\alpha(\pi-\alpha)}. \quad (3.6)$$

Here  $r_y(1) = \text{Im}[z(1)]$  is the distance of the tip of needle  $I$  from needle  $II$ , and we have used the relation  $r_y(1) = \pi^2 \mathcal{A}(\sin \alpha) / [\alpha(\pi - \alpha)]$ , which follows from Eq. (3.3).

As expected,  $f_y$  in Eq. (3.6) is an even function of the deviation  $\gamma = \alpha - \frac{1}{2}\pi$  from perpendicular orientation of the needles and diverges in the limit  $\gamma \rightarrow \pm \frac{1}{2}\pi$ , corresponding to parallel needles. From the values of  $\tilde{t}$  in Eq. (3.4), it follows that the force between the needles is attractive for  $OO$  and  $++$  boundaries and repulsive for  $+-$  and  $O+$  boundaries, with the strongest force in the  $+-$  case.

## B. Force between two semi-infinite needles

The conformal transformation  $w(z)$ , where

$$z'(w) = \mathcal{B} e^{i\alpha} w^{-\alpha/\pi-1} (w-1)(w+b), \quad (3.7)$$

$$z(w) = \pi \mathcal{B} \left[ e^{i\alpha} w^{-\alpha/\pi+1} \left( \frac{w}{2\pi-\alpha} + \frac{b-1}{\pi-\alpha} + \frac{b}{\alpha w} \right) - b^{-\alpha/\pi+1} \left( \frac{b}{2\pi-\alpha} - \frac{b-1}{\pi-\alpha} + \frac{1}{\alpha} \right) \right], \quad (3.8)$$

where  $\mathcal{B}$  and  $b$  are positive real constants and  $0 < \alpha < \pi$ , considered in §12.3 of Kober [36], maps the upper half  $w$  plane, with semi-infinite needles along the positive and negative  $u$  axes, onto the full  $z$  plane with two embedded semi-infinite needles. Needle  $II$ , the image of the negative  $u$  axis, coincides with the positive  $x$  axis. Needle  $I$ , the image of the positive  $u$  axis, extends from the point

$$z(1) = -\mathcal{B} \pi^2 \frac{[\alpha - b(2\pi - \alpha)] e^{i\alpha} + b^{-\alpha/\pi+1} (2\pi - \alpha - b\alpha)}{\alpha(\pi - \alpha)(2\pi - \alpha)} \quad (3.9)$$

to  $\infty$ , forming an angle  $\alpha$  with needle  $II$ .

The integrand in expressions (2.3), (2.4), and (2.5) for the Casimir force follows from Eqs. (2.2), (3.7), and (3.4) and is given by

$$\begin{aligned} z'(w) \langle T(z) \rangle = & \frac{e^{-i\alpha}}{48\pi^2 \mathcal{B}} \frac{w^{\alpha/\pi-1}}{(w-1)^3 (w+b)^3} \left\{ (3\pi - \alpha)(\pi - \alpha) w^4 + 2(1-b)\alpha(3\pi - \alpha) w^3 \right. \\ & + \left[ -\alpha(2\pi - \alpha) + 2b(5\pi^2 + 4\pi\alpha - 2\alpha^2) - b^2\alpha(2\pi - \alpha) \right] w^2 \\ & - 2b(1-b)(2\pi - \alpha)(\pi + \alpha)w - b^2(\pi^2 - \alpha^2) \\ & \left. + 48\pi^2 \tilde{t}(w-1)^2 (w+b)^2 \right\}. \end{aligned} \quad (3.10)$$



Integrating along path  $\mathcal{C}$  in Eq. (2.4) again amounts to integrating along the  $u$  axis from  $\rho_u = 0$  to  $+\infty$  in the  $w$  plane, passing above the pole at  $w = 1$ . Combining Eqs. (2.4) and (3.10), evaluating the integral, and making use of Eq. (2.3), we obtain [38]

$$\begin{aligned} \tau = -\frac{f_y + if_x}{k_B T} &= \frac{1}{96\pi^2 \mathcal{B}(1+b)^3 \sin \alpha} \times \\ &\times \left\{ \left[ \alpha(3\pi - \alpha) + 2(3\pi - \alpha)(\pi + \alpha)b + (2\pi - \alpha)(\pi + \alpha)b^2 - 96\pi^2(1+b)^2 \tilde{t} \right] + e^{-i\alpha} b^{\alpha/\pi-1} \right. \\ &\times \left. \left[ (2\pi - \alpha)(\pi + \alpha) + 2(3\pi - \alpha)(\pi + \alpha)b + \alpha(3\pi - \alpha)b^2 - 96\pi^2(1+b)^2 \tilde{t} \right] \right\}, \end{aligned} \quad (3.11)$$

where the parameters  $\mathcal{B}$  and  $b$  are related to the endpoint  $r_x(1), r_y(1)$  of needle  $I$  by Eq. (3.9).

Since needle  $II$  corresponds to the positive  $x$  axis, one expects to recover the results of the preceding subsection, in which needle  $II$  is infinite, in the limit  $r_x(1) \rightarrow +\infty$  with  $r_y(1)$  and  $\alpha$  fixed. According to Eq. (3.9) this limit is achieved on substituting  $\mathcal{B} = \mathcal{A}/b$  in the equation and then taking the limit  $b \rightarrow \infty$  with  $\mathcal{A}$  fixed. In this limit the derivative (3.7) reduces to Eq. (3.1), the integrand (3.10) reduces to (3.5), and the Casimir force (3.11) reduces to (3.6).

## IV. INTERACTIONS OF NEEDLES OF FINITE LENGTH

In this section the approach for infinite and semi-infinite needles is extended to needles of finite length. The region outside two finite needles, which is doubly connected, is generated from an annulus bounded by two concentric circles, for which the thermal average of the stress tensor is known.

### A. Two needles of finite length

#### 1. Conformal mapping

An arbitrary configuration of two non-overlapping needles  $I$  and  $II$  with finite lengths  $D_I$  and  $D_{II}$  in the  $z$  plane can be generated by a conformal transformation  $z(w)$  of the Schwarz-Christoffel type which maps the interior of the annulus  $h < |w| < 1$  in the  $w \equiv |w|e^{i\varphi}$  plane onto the region outside the two needles in the  $z$  plane. The desired mapping is a special case of the mapping onto the region outside two non-overlapping polygons derived by Akhiezer in 1928 and given at the end of §48 of Ref. [39]. In the special case in which the polygons reduce to needles, the mapping  $z(w)$  has the derivative

$$z'(w) = \frac{\mu}{w^2} \frac{\prod_{\ell=1}^4 \vartheta_1((2\pi i)^{-1} \ln(w/a_\ell))}{\vartheta_1^2((2\pi i)^{-1} \ln(w/c)) \vartheta_1^2((2\pi i)^{-1} \ln(wc))}, \quad (4.1)$$

in terms of the elliptic theta functions  $\vartheta_1$  and constants  $\mu$ ,  $a_\ell$ , and  $c$  defined in [39].

Substituting  $a_\ell = w_\ell$ ,  $c = Ch^{1/2}$ ,  $\mu = (w_1 w_2 w_3 w_4)^{-1/2} A$  in Eq. (4.1), and using the expression for  $\vartheta_1$  in Table IX of [39], we obtain the useful product representation

$$z'(w) = \frac{A}{w^2} \times \prod_{k=1}^{\infty} \frac{\prod_{\ell=1}^4 (1 - h^{2k-2} w/w_\ell)(1 - h^{2k} w_\ell/w)}{(1 - h^{2k-5/2} w/C)^2 (1 - h^{2k+1/2} C/w)^2 (1 - h^{2k-3/2} wC)^2 (1 - h^{2k-1/2}/(wC))^2}. \quad (4.2)$$

We will see that  $h \ll 1$  for needles short in comparison with their separation, and in this regime the representation (4.2) is especially convenient [40].

The complex constant  $A$  in Eq. (4.2) corresponds to a homogeneous rotation and dilatation, the positive real constant  $C$ , with  $h^{1/2} < C < h^{-1/2}$ , characterizes the value  $Ch^{1/2}$  of  $w$  which is mapped to  $z = \infty$ , and the points  $w_1 = e^{i\varphi_1}$ ,  $w_2 = e^{i\varphi_2}$  and  $w_3 = he^{i\varphi_3}$ ,  $w_4 = he^{i\varphi_4}$

on the outer and inner boundary circles of the annulus are the pre-images of the endpoints  $z_1, z_2$  and  $z_3, z_4$  of needles  $I$  and  $II$ , respectively. This is evident from the changes  $dz$  corresponding to displacements  $dw = d(e^{i\varphi})$  and  $dw = h d(e^{i\varphi})$  along the outer and inner boundaries of the annulus, for which Eq. (4.2) implies

$$dz = \frac{iA}{h} C^2 e^{-i(\varphi_3+\varphi_4)} \mathcal{G}(\varphi; \varphi_1, \varphi_2) \mathcal{P}(\varphi; \varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h) d\varphi \quad (4.3)$$

and

$$dz = \frac{iA}{h} \mathcal{G}(\varphi; \varphi_3, \varphi_4) \mathcal{P}(\varphi; \varphi_3, \varphi_4; \varphi_1, \varphi_2; C^{-1}; h) d\varphi, \quad (4.4)$$

respectively. Here

$$\begin{aligned} \mathcal{G}(\varphi; \varphi_J, \varphi_K) &= e^{-i\varphi} (1 - e^{i(\varphi-\varphi_J)}) (1 - e^{i(\varphi-\varphi_K)}) \\ &= -4 e^{-i(\varphi_J+\varphi_K)/2} \sin \frac{\varphi - \varphi_J}{2} \sin \frac{\varphi - \varphi_K}{2} \end{aligned} \quad (4.5)$$

and

$$\mathcal{P}(\varphi; \varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h) \equiv \prod_{k=1}^{\infty} \frac{\prod_{n=1,2} |1 - h^{2k} e^{i(\varphi-\varphi_n)}|^2 \prod_{m=3,4} |1 - h^{2k-1} e^{i(\varphi-\varphi_m)}|^2}{|1 - h^{2k-1/2} C^{-1} e^{i\varphi}|^4 |1 - h^{2k-3/2} C e^{i\varphi}|^4}, \quad (4.6)$$

which is always positive.

As  $\varphi$  varies, the argument of  $dz$  in Eq. (4.3) stays constant except at  $\varphi = \varphi_1$  and  $\varphi = \varphi_2$  where, due to the factor  $\mathcal{G}(\varphi; \varphi_1, \varphi_2)$ , it changes by  $\pi$ . This corresponds to constant slopes along the two sides of needle  $I$  with endpoints  $z_1$  and  $z_2$ . Analogous results for needle  $II$  follow from Eq. (4.4).

Moreover, moving counterclockwise inside the annulus close to the outer and inner boundary circle corresponds to encircling needle  $I$  clockwise and needle  $II$  counterclockwise, respectively, in the  $z$  plane. This is most easily verified near the needle tips  $z_\ell$ , where, due to Eq. (4.2),  $dz \equiv z'(w)dw = \text{const} \times (w - w_\ell)dw$ , since  $w - w_\ell$  turns 180 degrees clockwise and counterclockwise, respectively, on passing point  $w_\ell$  on the outer and inner boundary.

Without loss of generality and for later convenience we assume

$$-\pi < \varphi_\ell \leq \pi; \quad \ell = 1, 2, 3, 4 \quad (4.7)$$

for the arguments of the four pre-images  $w_\ell$  of the needle ends.

The mapping  $z = z(w)$  is required to be single-valued, so that the displacement  $z(w_a) - z(w_b) = \int_{w_b}^{w_a} (dz/dw)dw$  for any two points  $w_a$  and  $w_b$  in the annulus is independent of the

integration path. For  $w_a = w_b$  the integral must vanish, even if the path encloses the inner boundary circle or the singularity at  $w = Ch^{1/2}$ . To ensure this, we require that the integrals

$$\int_{\varphi=-\pi}^{\varphi=\pi} \left( \frac{dz}{dw} dw \right)_{w=e^{i\varphi}} = 0, \quad \int_{\varphi=-\pi}^{\varphi=\pi} \left( \frac{dz}{dw} dw \right)_{w=he^{i\varphi}} = 0 \quad (4.8)$$

around the outer and inner boundary circles vanish. On inserting (4.3)-(4.6) in (4.8) and discarding  $\varphi$ -independent complex factors, the conditions (4.8) imply the vanishing of two real functions of the six real parameters  $\varphi_1, \dots, \varphi_4, C, h$ . This leaves four independent parameters (apart from the complex constant  $A$ ), consistent with the four degrees of freedom needed to specify (apart from homogeneous translations, rotations, and dilatations) a configuration of two needles, for example, the two lengths and the two angles the needles form with the vector between their midpoints. According to Eqs. (4.3) and (4.4), the second of the conditions (4.8) follows from the first on exchanging the pairs  $\varphi_1, \varphi_2$  and  $\varphi_3, \varphi_4$  and replacing  $C$  by  $C^{-1}$ . This leads from one allowed parameter set to another. Explicit expressions for small  $h$  are given in Eqs. (4.16) below.

In conformity with above remarks, we use the notation

$$\begin{aligned} z_1 - z_2 = z_{12} = |z_{12}| e^{i\Phi_{12}} &\equiv D_I e^{i\Phi_I}, \quad z_3 - z_4 = z_{34} = |z_{34}| e^{i\Phi_{34}} \equiv D_{II} e^{i\Phi_{II}}, \\ (z_1 + z_2)/2 = z_I, \quad (z_3 + z_4)/2 = z_{II}, \end{aligned} \quad (4.9)$$

and

$$z_I - z_{II} = z_{I,II} \quad (4.10)$$

for the needle vectors, the positions of their midpoints, and their separation vector.

The needle vectors  $z_{12}$  and  $z_{34}$  follow from the first and second integrals in Eq. (4.8) on replacing the lower and upper limits  $-\pi$  and  $\pi$  by  $\varphi_2$  and  $\varphi_1$  and by  $\varphi_4$  and  $\varphi_3$ , respectively. This yields

$$\begin{aligned} z_{12} &= e^{-i(\varphi_3 + \varphi_4)} (iA/h) \mathcal{CN}(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h), \\ z_{34} &= (iA/h) \mathcal{CN}(\varphi_3, \varphi_4; \varphi_1, \varphi_2; C^{-1}; h), \end{aligned} \quad (4.11)$$

where, on using Eqs. (4.5), (4.6),

$$\mathcal{N}(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h) = e^{-i(\varphi_1 + \varphi_2)/2} \operatorname{sgn}(\varphi_1 - \varphi_2) P(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h). \quad (4.12)$$

Here

$$\begin{aligned}
P(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h) &= \\
&= 4C \left| \int_{\varphi_2}^{\varphi_1} d\varphi \sin \frac{\varphi - \varphi_1}{2} \sin \frac{\varphi - \varphi_2}{2} \mathcal{P}(\varphi; \varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h) \right|, \quad (4.13)
\end{aligned}$$

where  $\mathcal{P}$  is given in Eq. (4.6).

For the angle enclosed by the two needles, Eqs. (4.11)-(4.13) imply the simple relation

$$e^{i(\Phi_{12} - \Phi_{34})} = e^{-i(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/2} \operatorname{sgn}(\varphi_1 - \varphi_2) \operatorname{sgn}(\varphi_3 - \varphi_4). \quad (4.14)$$

Note that in the sector (4.7) the complex numbers  $e^{-i(\varphi_1 + \varphi_2)/2}$  and  $e^{-i(\varphi_3 + \varphi_4)/2}$  and the signs of  $\varphi_1 - \varphi_2$  and  $\varphi_3 - \varphi_4$  are uniquely determined by  $w_1, w_2$  and by  $w_3, w_4$ , respectively.

For the ratio of needle lengths Eqs. (4.11)-(4.13) yield

$$D_I/D_{II} \equiv |z_{12}|/|z_{34}| = P(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h)/P(\varphi_3, \varphi_4; \varphi_1, \varphi_2; C^{-1}; h), \quad (4.15)$$

so that exchanging the pairs  $\varphi_1, \varphi_2$  and  $\varphi_3, \varphi_4$  and replacing  $C$  by  $C^{-1}$  changes  $D_I/D_{II}$  to its inverse. For the special parameter sets  $C = 1$  with either  $\varphi_3 = \varphi_1, \varphi_4 = \varphi_2$  or  $\varphi_3 = -\varphi_1, \varphi_4 = -\varphi_2$ , the two needles have equal lengths  $|z_{12}| = |z_{34}|$ . For the second set this follows from Eq. (4.6), which implies  $\mathcal{P}(\varphi; \varphi_1, \varphi_2; -\varphi_1, -\varphi_2; C; h) = \mathcal{P}(-\varphi; -\varphi_1, -\varphi_2; \varphi_1, \varphi_2; C; h)$ , yielding  $P(\varphi_1, \varphi_2; -\varphi_1, -\varphi_2; 1; h) = P(-\varphi_1, -\varphi_2; \varphi_1, \varphi_2; 1; h)$ .

We mention another, rather obvious, property of the transformation (4.2): Changing the parameters from  $(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C; h)$  to  $(-\varphi_1, -\varphi_2; -\varphi_3, -\varphi_4; C; h)$ , i.e., changing all four  $w_\ell$  to  $w_\ell^*$ , leads from one single-valued mapping to another, in which the needle configuration is changed from  $(z_{12}, z_{34}; z_{I,II})$  to  $(z_{12}^*, z_{34}^*; z_{I,II}^*)$ , assuming  $A$  is real. Here and below an asterisk denotes complex conjugation.

Except for the enclosed angle it is, in general, not obvious how to choose the parameters in the transformation (4.2) to generate a given configuration of the two needles. Here we list some simple classes (A)-(E) of needle configurations which only require a parameter search in a reduced subspace. We choose the vector between the needle centers to be parallel to the real axis, so that  $z_{I,II} = |z_{I,II}|$ , with needle  $I$  to the right of needle  $II$ , and refer to the ratios  $|z_{12}|/|z_{I,II}|$  and  $|z_{34}|/|z_{I,II}|$  as the “reduced needle lengths”.

(A) Symmetric-perpendicular configurations of two needles of arbitrary reduced lengths with the symmetry of the letter T, corresponding to Fig. 1-(A): To be specific, we consider

the needle vectors  $z_{12} = i|z_{12}|$  and  $z_{34} = -|z_{34}|$ . These configurations can be generated from parameters in the subspace  $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (-|\varphi_1|, |\varphi_1|, 0, \pi)$  where  $w_2 = w_1^*$ ,  $w_3 = h$ ,  $w_4 = -h$ . The reason is that on choosing  $A$  real, the integrals over (4.2) from  $w = w_2$  to  $w = w_3$  and from  $w = w_1$  to  $w = w_3$  (and likewise those from  $w = w_2$  to  $w = w_4$  and from  $w = w_1$  to  $w = w_4$ ) are complex conjugates, implying the properties  $z_2 - z_3 = (z_1 - z_3)^*$  and  $z_2 - z_4 = (z_1 - z_4)^*$ . Since the expression multiplying  $d\varphi$  in Eq. (4.4) is an odd function of  $\varphi$ , the integral in Eq. (4.8) around the inner circle vanishes for all values of  $\varphi_1$ ,  $C$ , and  $h$ . The requirement that the integral over the outer circle vanish implies a relation  $\varphi_1 = \psi(C, h)$ , leaving  $C$  and  $h$  free to generate given values for the two reduced needle lengths. Note, finally, that the general enclosed angle relation (4.14) is satisfied, since both of its sides equal  $-i$  in the above subspace of parameters in the annulus and for the needle configuration in which  $e^{i\Phi_{12}} = i$ ,  $e^{i\Phi_{34}} = -1$ .

(B) Symmetric-parallel configurations of two needles with arbitrary reduced lengths perpendicular to the vector between their centers, corresponding to Fig. 1-(B): To be specific we choose  $z_{12} = i|z_{12}|$  and  $z_{34} = i|z_{34}|$ , so that the needles are parallel to the imaginary axis. These configurations are generated by  $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (-|\varphi_1|, |\varphi_1|, -|\varphi_3|, |\varphi_3|)$ , i.e., by  $w_2 = w_1^*$ ,  $w_4 = w_3^*$  since, as in case (A), for real values of the parameter  $A$  there is reflection symmetry about the real axis, and the integrals over (4.2) from  $w = w_2$  to  $w = w_3$  and from  $w = w_1$  to  $w = w_4$  (and likewise those from  $w = w_2$  to  $w = w_4$  and from  $w = w_1$  to  $w = w_3$ ) are complex conjugates, implying the properties  $z_2 - z_3 = (z_1 - z_4)^*$  and  $z_2 - z_4 = (z_1 - z_3)^*$ . The vanishing of the two integrals (4.8) implies relations  $\varphi_1 = \chi(C, h)$  and  $\varphi_3 = \omega(C, h)$ , and the two parameters  $C$  and  $h$  can be adjusted to generate the two reduced needle lengths. The enclosed angle relation (4.14) is satisfied, with both sides equal to 1.

(C) Configurations of two needles which are mirror symmetric about the imaginary axis, corresponding to Fig. 1-(C): Here  $z_{12} = |z_{12}|e^{i\Phi_{12}}$ ,  $z_{34} = |z_{12}|e^{i(\pi-\Phi_{12})} = -z_{12}^*$ . The needles have the same arbitrary reduced length, and with no loss of generality the angle  $\pi - 2\Phi_{12}$  between them can be restricted to values between 0 and  $\pi$ .

(D) Antiparallel configurations of two needles with the same arbitrary reduced length, corresponding to Fig. 1-(D): Here  $z_{12} = |z_{12}|e^{i\Phi_{12}} = -z_{34}$ , where the angle  $\Phi_{12}$  is arbitrary [41].

The needle configurations (C) and (D) are generated by  $C = 1$  in both cases and by  $\varphi_3 = \varphi_1$ ,  $\varphi_4 = \varphi_2$  in case (C) and  $\varphi_3 = -\varphi_1$ ,  $\varphi_4 = -\varphi_2$  in case (D). In both subspaces the

lengths  $|z_{12}|$ ,  $|z_{34}|$  of the two needles are equal for arbitrary values of the three parameters  $\varphi_1$ ,  $\varphi_2$ , and  $h$ , and the two conditions in (4.8) reduce to a single condition (see the remarks below Eqs. (4.15) and (4.8), respectively). This leaves two free parameters, which can be adjusted to generate the given common reduced length of the needles and the angle  $\Phi_{12}$  needle  $I$  forms with the distance vector  $z_{I,II}$  between the needle midpoints. In case (C) the enclosed angle relation (4.14) predicts  $e^{2i\Phi_{12}} = -e^{-i(\varphi_1+\varphi_2)}$ , and in case (D) it is satisfied since both sides equal  $-1$ .

Typical configurations from classes (A)-(D) are shown in Fig. 1. Classes (B), (C), and (D) encompass two particularly simple needle configurations for which the conformal mapping can be found in the literature:

(i) Collinear needles of equal length with  $\Phi_{12} = 0$  and  $\Phi_{34} = \pi$  are generated by  $(w_1, w_2, w_3, w_4) = (1, -1, h, -h)$  and  $C = 1$ , which is a special case of both (C) and (D). In this case the two conditions (4.8) are satisfied, since both integrands (4.3) and (4.4) are odd functions of  $\varphi$ . The reduced needle length is determined by the parameter  $h$ . The corresponding conformal transformation  $z(w)$  is discussed in Refs. [36, 39].

(ii) Two parallel needles of equal length with  $\Phi_{12} = \Phi_{34} = \pi/2$ , i.e., a configuration with the symmetry of the letter H: This needle geometry is generated by  $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (-|\varphi_1|, |\varphi_1|, -|\varphi_1|, |\varphi_1|)$  and  $C = 1$ , which is a special case of both (B) and (C). The two parameters  $|\varphi_1|$  and  $h$  are chosen to satisfy the two *identical* conditions (4.8) and to generate a given reduced needle length. The conformal transformation leading to this needle configuration is considered in some detail in Refs. [36, 39].

(E) Widely separated needles: Needles with lengths  $|z_{12}|$ ,  $|z_{34}|$  much smaller than their separation  $|z_{I,II}|$  are generated by Eq. (4.2) on choosing  $C$  of order 1 and  $h \ll 1$ . While detailed results for the mapping in cases (A)-(D) for needles of arbitrary length can only be obtained numerically (see Section V), for widely separated needles analytic results may be derived by expanding in terms of the small parameter  $h^{1/2}$ . Using the two conditions (4.8) to express  $\varphi_2$  and  $\varphi_4$  in terms of the four free parameters  $\varphi_1$ ,  $\varphi_3$ ,  $C$ , and  $h$  leads to

$$\begin{aligned}\varphi_2 &= \varphi_1 - \pi \operatorname{sgn}(\varphi_1 - \varphi_2) + G(\varphi_1; \varphi_3; C; h), \\ \varphi_4 &= \varphi_3 - \pi \operatorname{sgn}(\varphi_3 - \varphi_4) + G(\varphi_3; \varphi_1; C^{-1}; h),\end{aligned}\tag{4.16}$$

where

$$G(\varphi_1; \varphi_3; C; h) = 4h^{1/2}C \sin \varphi_1 + 4hC^2 \sin(2\varphi_1) -$$

$$-4h^{3/2}\left\{\frac{1}{3}C^3(7-16\cos^2\varphi_1)\sin\varphi_1+C^{-1}[2\sin\varphi_3\cos(\varphi_1-\varphi_3)-\sin\varphi_1]\right\}, \quad (4.17)$$

apart from terms of order  $h^2$ . Equations (4.16) reflect the symmetry mentioned below Eq. (4.8) and are consistent with our assumption (4.7). The dependence of the needle configuration on the four free parameters is given by

$$\frac{z_{12}}{z_{I,II}} = R(\varphi_1, \varphi_3, C, h), \quad \frac{z_{34}}{z_{I,II}} = -R(\varphi_3, \varphi_1, C^{-1}, h)^*, \quad (4.18)$$

where

$$R(\varphi_1, \varphi_3, C, h) = 4Ch^{1/2}e^{-i\varphi_1} \times \\ \times \left\{1 - 2ih^{1/2}C\sin\varphi_1 - h[C^{-2}(1+2e^{2i\varphi_3}) + C^2] + \mathcal{O}(h^{3/2})\right\}. \quad (4.19)$$

The symmetry embodied in Eqs. (4.18), which we have checked within the  $h$ -expansion, is expected to hold for arbitrary  $h$ . Starting from a set  $(\varphi_1, \varphi_2; \varphi_3, \varphi_4; C, h)$  of six parameters obeying the two conditions (4.8) and replacing it by  $(\varphi_3, \varphi_4; \varphi_1, \varphi_2; C^{-1}, h)$  corresponds to reflecting the two needle configuration about the symmetry axis of the needle midpoints, i.e., about the imaginary axis of the  $z$ -plane if we choose  $z_I = |z_I|$  and  $z_{II} = -|z_I|$  by adjusting  $A$  appropriately. This is consistent with the above discussion of the reflection-invariant needle configuration (C), for which  $C = 1$ .

For the ratio of needle vectors, Eqs. (4.18) and (4.19) imply

$$\frac{z_{12}}{z_{34}} = C^2 \left[1 + 2h \left(C^2 \cos^2 \varphi_1 - C^{-2} \cos^2 \varphi_3\right)\right] \times \\ \times \exp \left(-i \left\{\varphi_1 + \varphi_3 + \pi + \frac{1}{2}[G(\varphi_1; \varphi_3; C; h) + G(\varphi_3; \varphi_1; C^{-1}; h)]\right\}\right) + \\ + \mathcal{O}(h^{3/2}), \quad (4.20)$$

where the phase factor and modulus are consistent with the enclosed angle relation (4.14) and the ratio of needle lengths (4.15), respectively. We also note the relation

$$z_{I,II} = A h^{-3/2} C e^{-2i\varphi_3} \left\{1 + 2h^{1/2} C^{-1} (-e^{i\varphi_3} + e^{-i\varphi_3}) + \right. \\ \left. + h[C^{-2}(2e^{2i\varphi_3} - 3 + 4e^{-2i\varphi_3}) + C^2(1 + 2e^{-2i\varphi_1})] + \mathcal{O}(h^{3/2})\right\}, \quad (4.21)$$

which determines the value of  $A$  needed to generate a given  $z_{I,II} = |z_{I,II}|$ .

## 2. Force and torque

The force and torque on needle  $I$  due to needle  $II$  can be evaluated using Eqs. (2.2)-(2.5) and (2.7)-(2.11), respectively. The stress tensor average in the annulus [42] was determined



by Cardy [31] and can be written as

$$\langle T(w) \rangle \equiv \langle T(w) \rangle_{\text{annulus}} = \frac{1}{2w^2} t(h), \quad (4.22)$$

where

$$t(h) \equiv h \frac{d}{dh} \ln \left\{ \left( 1 + [(\mathcal{S}_{11} + \mathcal{S}_{21})/2, \mathcal{S}_{11}, \mathcal{S}_{21}, \mathcal{S}_{22}] \right) \prod_{n=1}^{\infty} (1 - h^{2n})^{-1} \right\} \quad (4.23)$$

for the combinations  $OO$ ,  $++$ ,  $+-$ ,  $O+$  of universality classes of the two needles [43]. Here

$$\mathcal{S}_{pq} \equiv \mathcal{S}_{pq}(h) = \sum_{r=2}^{\infty} h^{(r^2-1)/24} \sin \frac{\pi p r}{3} \sin \frac{\pi q r}{4} / \left( \sin \frac{\pi p}{3} \sin \frac{\pi q}{4} \right), \quad (4.24)$$

where the series converges for  $h < 1$ . For the integration path  $\mathcal{C}$  in (2.5) and (2.11), which goes around the inner boundary circle counterclockwise, it is most convenient to use the circle  $\mathcal{C}_c$  given by  $w = Ch^{1/2}e^{i\varphi}$ , which passes through the pre-image  $w = Ch^{1/2}$  of  $z = \infty$ .

Unlike the force and torque contributions  $\tau^{(T)}$  and  $\theta^{(T)}$  in (2.5) and (2.11), which depend, via Eqs. (4.22)-(4.24), on the surface universality classes of the two needles, the contributions  $\tau^{(S)}$  and  $\theta^{(S)}$ , which involve  $z'(w)$  and the Schwarzian derivative (2.2), are solely determined by the geometric configuration of the needles and in this sense “hyper-universal”. This was already mentioned at the end of Sec. II, and it applies to the semi-infinite and infinite needles of Sec. III. The occurrence of a hyper-universal term in the free energy of interaction of a non-circular particle with other particles in a near-critical two-dimensional system is well known from the “small-particle operator expansion” (SPOE). As discussed in Refs. [12, 44] and Appendix B 2 below, the hyper-universal interaction arises from the stress tensor in the operator expansion corresponding to the particle, in our case a needle. The hyper-universal term in the expansion depends on the orientation of the needle, is proportional to the square of its length, which is the smallest power involving its orientation dependence, and it reproduces the results corresponding to the  $h \rightarrow 0$  contributions of  $\tau^{(S)}$  and  $\theta^{(S)}$ , as we show in Eqs. (4.29)-(4.31)

In general, the force vector is neither parallel nor antiparallel to the vector  $z_{I,II}$  between the needle midpoints, as seen, for example, in Eq. (4.29) below. However, for the symmetric perpendicular and parallel configurations in (A) and (B) and for the mirror symmetric configurations (C), the force clearly points along  $z_{I,II}$  or  $-z_{I,II}$ . Detailed numerical results for force and torque in cases (A)-(D) are reported in Sec. V. Here we give a few analytic results for the case (E) of two short needles or, equivalently, two widely separated needles.

According to Eqs. (4.18), (4.19), this regime corresponds to small  $h$ , and in leading order

$$\tau^{(T)} = \frac{2i}{z_{I,II}} t(h \ll 1), \quad \tau^{(S)} = \frac{2i}{z_{I,II}} h^2 e^{2i(-\varphi_1 + \varphi_3)}, \quad (4.25)$$

for the two contributions to the force on needle  $I$  in (2.3). For more details see Eqs. (B1)-(B6) in Appendix B. On using (4.23) and (4.24), we obtain

$$t(h \ll 1) = [h, -h] \quad (4.26)$$

for needle pairs of type  $[OO, O+]$ , while for pairs of type  $[++, +-]$

$$t(h \ll 1) = \frac{1}{8} \left[ \sqrt{2} h^{1/8} / \left( 1 + \sqrt{2} h^{1/8} \right), \quad -\sqrt{2} h^{1/8} / \left( 1 - \sqrt{2} h^{1/8} \right) \right], \quad (4.27)$$

for  $h$  values which are small compared to 1 without requiring  $h^{1/8}$  to be small. The force  $(f_x, f_y)$  on needle  $I$  in (2.3) is thus dominated by the contribution from  $\tau^{(T)}$ . In the remainder of this subsection we again assume that  $z_{I,II} \equiv |z_{I,II}|$ , so that the distance vector  $z_{I,II}$  between needle centers is parallel to the  $x$ -axis. Using  $h \rightarrow |z_{12}| |z_{34}| / (16 |z_{I,II}^2|)$  due to (4.18), (4.19), one finds

$$f_x / k_B T = - \frac{1}{|z_{I,II}|} \times \left[ \pm \frac{1}{8} \frac{D_I D_{II}}{|z_{I,II}|^2}, \pm \frac{1}{4} \frac{(D_I D_{II})^{1/8}}{|z_{I,II}|^{1/4}} / \left( 1 \pm \frac{(D_I D_{II})^{1/8}}{|z_{I,II}|^{1/4}} \right) \right] \quad (4.28)$$

and  $f_y / (k_B T) = 0$  for the force components in leading order. Here the upper and lower signs describe the needle universality classes  $[OO, ++]$  and  $[O+, +-]$ , respectively, and  $D_I \equiv |z_{12}|$  and  $D_{II} \equiv |z_{34}|$  are the needle lengths introduced in Eq. (4.9). For needles with equal (unequal) universality classes the force is antiparallel (parallel) to the distance vector  $z_{I,II}$ , i.e., attractive (repulsive), as expected. As in a multi-pole expansion, the shape anisotropy does not appear in the leading “monopole” contribution (4.28), in which the force is independent of the needle orientations  $\Phi_{12}$  and  $\Phi_{34}$ , but it appears in higher order in the needle lengths. Unlike the corresponding higher order contributions from  $\tau^{(T)}$ , which also depend on the needle universality classes, the contributions from  $\tau^{(S)}$  are hyper-universal, as mentioned above. Equations (4.18) and (4.19) imply  $e^{-i\varphi_1} \rightarrow e^{i\Phi_{12}}$  and  $e^{i\varphi_3} \rightarrow -e^{i\Phi_{34}}$ , and the lowest-order hyperuniversal term, given by the second expression in (4.25), leads to

$$(f_x^{(S)}, f_y^{(S)}) / (k_B T) = - \frac{D_I^2 D_{II}^2}{2^7 |z_{I,II}|^5} \left( \cos(2(\Phi_{12} + \Phi_{34})), -\sin(2(\Phi_{12} + \Phi_{34})) \right). \quad (4.29)$$

As expected, the force is unchanged on rotating a needle through 180 degrees.

We now turn from Eq. (4.29) to the hyperuniversal contribution  $-\text{Re } \theta^{(S)}$  to the reduced torque  $\Theta/(k_B T)$ , introduced in Eqs. (2.7)-(2.11). Calculating  $\theta^{(S)}$  by means of the mapping (4.2), one obtains

$$\begin{aligned}\theta^{(S)} &\rightarrow -i h^2 e^{2i(-\varphi_1 + \varphi_3)}, \\ \text{Re } \theta^{(S)} &\rightarrow \frac{D_I^2 D_{II}^2}{2^8 |z_{I,II}|^4} \sin(2(\Phi_{12} + \Phi_{34}))\end{aligned}\quad (4.30)$$

in leading order. For more details see the paragraph containing Eqs. (B7), (B8) in Appendix B.

A detailed discussion of the force and torque for two short needles, based on the SPOE, is given in Appendix B 2 a. With this entirely different approach we confirm the leading behavior (4.28) for the force and obtain

$$\delta F^{(\text{hu})}/(k_B T) = -\frac{D_I^2 D_{II}^2}{2^{10}} \left( e^{2i(\Phi_{12} + \Phi_{34})} \frac{1}{z_{I,II}^4} + \text{c.c.} \right) \quad (4.31)$$

for the hyperuniversal (hu) contribution to the free energy of interaction [35] of the needles, which agrees with the results for the force components in (4.29) and the torque in (4.30).

## B. Interaction of a finite and a semi-infinite needle

Consider the case in which needle 12 has a finite length  $D_I \equiv |z_{12}|$  but needle 34 is semi-infinite, with  $z_3 = z_e$  and  $z_4 = \infty$ . This needle geometry is generated by Eq. (4.2) in the limit  $C = h^{1/2}$  and  $w_4 = h$  in which the pre-images  $w_4$  and  $Ch^{1/2}$  of  $z = z_4$  and  $z = \infty$ , respectively, coincide, so that

$$\begin{aligned}z'(w) &= \frac{A}{h^2} \prod_{k=1}^{\infty} \left( \frac{(1 - h^{2k-3} e^{-i\varphi_e} w)(1 - h^{2k+1} e^{i\varphi_e} / w)}{(1 - h^{2k-3} w)^3 (1 - h^{2k+1} / w)^3} \times \right. \\ &\quad \left. \times \prod_{n=1,2} (1 - h^{2k-2} e^{-i\varphi_n} w)(1 - h^{2k} e^{i\varphi_n} / w) \right).\end{aligned}\quad (4.32)$$

This implies

$$\begin{aligned}dz &= (d\varphi) i A e^{-i\varphi_e} \mathcal{G}(\varphi; \varphi_1, \varphi_2) \times \\ &\quad \times \prod_{k=1}^{\infty} \frac{|1 - h^{2k-1} e^{i(\varphi - \varphi_e)}|^2 \prod_{n=1,2} |1 - h^{2k} e^{i(\varphi - \varphi_n)}|^2}{|1 - h^{2k-1} e^{i\varphi}|^6}\end{aligned}\quad (4.33)$$

and

$$dz = -(d\varphi) \frac{iA}{h} e^{-i\varphi_e/2} \frac{\sin[(\varphi - \varphi_e)/2]}{4 \sin^3(\varphi/2)} \times \\ \times \prod_{k=1}^{\infty} \frac{|1 - h^{2k} e^{i(\varphi - \varphi_e)}|^2 \prod_{n=1,2} |1 - h^{2k-1} e^{i(\varphi - \varphi_n)}|^2}{|1 - h^{2k} e^{i\varphi}|^6}, \quad (4.34)$$

for displacements  $dw = d(e^{i\varphi})$  and  $dw = h d(e^{i\varphi})$  around the outer and inner boundaries of the annulus, respectively. Here  $\mathcal{G}$  is defined in Eq. (4.5), and  $w_3 \equiv w_e = h e^{i\varphi_e}$  is the pre-image of  $z_e$ . The behavior (4.34) near  $\varphi = 0$  implies that the semi-infinite needle extends from  $z = z_e$  to  $z = s|\infty|$  along the tangential unit vector  $s = -i(A/|A|)e^{-i\varphi_e/2} \text{sgn } \varphi_e$  with  $\varphi_e \equiv \varphi_3$  obeying (4.7). For convenience we choose the semi-infinite needle to coincide with the positive real axis, i.e.,  $z_e = 0$  and  $s = 1$ , so that

$$A = i|A|e^{i\varphi_e/2} \text{sgn } \varphi_e. \quad (4.35)$$

From Eq. (4.14) we obtain

$$e^{i\Phi_{12}} = -e^{-i(\varphi_1 + \varphi_2 + \varphi_e)/2} \text{sgn } (\varphi_1 - \varphi_2) \text{sgn } \varphi_e, \quad (4.36)$$

since  $\Phi_{34} = \pi$ ,  $\varphi_4 = 0$ , and  $\varphi_3 = \varphi_e$ .

Apart from homogeneous dilatations, the needle configuration is determined by three parameters - the length ratio  $|z_{12}|/|z_I|$  and the two angles  $\arg z_I$  and  $\Phi_{12}$  which  $z_I$  and  $z_{12}$  form with the semi-infinite needle. Here  $z_I \equiv r_{I,x} + ir_{I,y}$  is the vector from  $z_e = 0$  to the midpoint of needle 12. Correspondingly, there are, apart from  $|A|$ , three independent mapping parameters. Since the derivative  $dz/dw$  is analytic in the interior of the annulus, imposing the requirement

$$I_{\text{outer}} \equiv \int_{\mathcal{C}_{\text{outer}}} dw(dz/dw) = 0 \quad (4.37)$$

on the four parameters  $h, \varphi_1, \varphi_2, \varphi_e$  ensures that the mapping  $z = z(w)$  is single valued.

Now consider the case of a finite needle which is much shorter than its distance from the closest point of the semi-infinite needle, so that  $|z_{12}|^2 \ll |z_I|(|z_I| - r_{I,x})$ . Explicit results for the force and torque in this regime can be obtained by expanding in terms of  $h$  and are expected to agree with the SPOE. In the remainder of this subsection this is checked in leading order.

For small  $h$  the constraint (4.37) reads

$$\varphi_2 = \varphi_1 - \pi \operatorname{sgn}(\varphi_1 - \varphi_2) + 2h[3 \sin \varphi_1 - \sin(\varphi_1 - \varphi_e)] + \mathcal{O}(h^2), \quad (4.38)$$

yielding in terms of independent parameters the needle configuration

$$z_I \rightarrow \frac{|A|}{4h|\sin \varphi_e/2|} e^{-i\varphi_e}, \quad (4.39)$$

$$z_{12} \rightarrow -4|A|ie^{-i\varphi_1} e^{-i\varphi_e/2} \operatorname{sgn} \varphi_e, \quad (4.40)$$

to leading order in  $h$ . Since  $ie^{-i\varphi_1}$  equals  $e^{-i(\varphi_1+\varphi_2)/2} \operatorname{sgn}(\varphi_1 - \varphi_2)$  in leading order, Eq. (4.40) is consistent with (4.36). Equations (4.39) and (4.40) allow us to express  $|A|$ ,  $h$ ,  $\varphi_1$ ,  $\varphi_e$  in terms of needle parameters, and we note that

$$|A| \rightarrow |z_{12}|/4 \quad (4.41)$$

and

$$h \rightarrow \frac{1}{8} \frac{|z_{12}|}{\sqrt{2|z_I|(|z_I| - r_{I,x})}} \quad (4.42)$$

for use below.

The force  $(f_x, f_y)$  on the 12 needle follows from Eqs. (2.3) and (2.5) on substituting the derivative (4.32), integrating along a circle infinitesimally larger than the inner boundary circle of the annulus, avoiding the singularity at  $w = w_e = he^{i\varphi_e}$ .

The leading contribution comes from  $\tau^{(T)}$  and is given by

$$\tau^{(T)}/t(h) \rightarrow \frac{1}{4|z_I|} \frac{e^{i\varphi_e/2}(e^{i\varphi_e} - 3)}{\sin(\varphi_e/2)}, \quad (4.43)$$

yielding

$$[(f_x - if_y)/(k_B T)]/t(h) \equiv i\tau/t(h) \rightarrow -\frac{3z_I - |z_I|}{2z_I(z_I - |z_I|)}, \quad (4.44)$$

since  $z_I/|z_I| \rightarrow e^{-i\varphi_e}$ , see Eq. (4.39). The leading contribution to the force follows from Eq. (4.44) on replacing  $t(h)$  by  $t(h \ll 1)$  in Eqs. (4.26), (4.27) and on replacing  $h$  by the right hand side of Eq. (4.42). As in Eq. (4.28), the leading contribution is independent of the orientation  $\Phi_{12}$  of the small needle. The leading dependence on orientation comes from  $\tau^{(S)}$  in (2.5) in higher order in  $h$ . In Appendix B 2 b we use the SPOE to calculate the leading isotropic and angle-dependent contributions to the force and the leading contribution to the torque on the 12 needle. The SPOE prediction is in complete agreement with the  $h$ -expansion result for the force given in Eq. (4.44).

### C. Interaction of a finite and an infinite needle

On setting  $w_3 = w_4 = Ch^{1/2} = h$  in Eq. (4.2), both  $z_3$  and  $z_4$  become infinite, so that the 34 needle takes the form of an infinite needle or boundary line. For  $A = -i|A|$ , the infinite needle coincides with the boundary  $\text{Im}z = 0$  of the upper half plane. The derivative of the transformation is given by

$$z'(w) = -i|A| \frac{1}{h^2} \prod_{k=1}^{\infty} \frac{\prod_{n=1,2} (1 - h^{2k-2} e^{-i\varphi_n} w)(1 - h^{2k} e^{i\varphi_n}/w)}{(1 - h^{2k-3} w)^2 (1 - h^{2k+1}/w)^2}, \quad (4.45)$$

so that

$$dz = (d\varphi) |A| \mathcal{G}(\varphi; \varphi_1, \varphi_2) \prod_{k=1}^{\infty} \frac{\prod_{n=1,2} |1 - h^{2k} e^{i(\varphi - \varphi_n)}|^2}{|1 - h^{2k-1} e^{i\varphi}|^4} \quad (4.46)$$

and

$$dz = -(d\varphi) \frac{|A|}{h} \frac{1}{4 \sin^2(\varphi/2)} \prod_{k=1}^{\infty} \frac{\prod_{n=1,2} |1 - h^{2k-1} e^{i(\varphi - \varphi_n)}|^2}{|1 - h^{2k} e^{i\varphi}|^4} \quad (4.47)$$

for displacements  $dw = d(e^{i\varphi})$  and  $dw = h d(e^{i\varphi})$  along the outer and inner boundary circles, respectively. Thus, a counterclockwise path around the inner circle corresponds to a path along the real axis from  $+\infty$  to  $-\infty$ , and a counterclockwise path around the outer circle to a clockwise path around the 12 needle along its edges. The function  $\mathcal{G}$  is defined by Eq. (4.5).

As in the case (4.32) of a finite needle interacting with a semi-infinite needle, the constraint  $I_{\text{outer}} = 0$  ensures that the mapping (4.45) is single-valued, so that only two of the three parameters  $h, \varphi_1, \varphi_2$  are independent. They can be adjusted to fix the angle  $\Phi_{12}$  between needle and boundary and the ratio of the length  $|z_{12}|$  of the needle and its distance to the boundary. According to the argument leading to Eq. (4.14), the unit vector characterizing the direction of  $z_{12}$  is given by

$$e^{i\Phi_{12}} \equiv \frac{z_{12}}{|z_{12}|} = e^{-i(\varphi_1 + \varphi_2)/2} \text{sgn}(\varphi_1 - \varphi_2). \quad (4.48)$$

#### 1. Distant needle

A needle far from the boundary in comparison with its length corresponds to  $h \ll 1$ . The products in (4.45)-(4.47) can be expanded in powers of  $h$ , and Eq. (4.37) yields

$$\varphi_2 = \varphi_1 - \pi \text{sgn}(\varphi_1 - \varphi_2) + g(\varphi_1; h),$$

$$g(\varphi_1; h) \equiv 4h \sin \varphi_1 + 4h^2 \sin(2\varphi_1) + \mathcal{O}(h^3), \quad (4.49)$$

which is consistent with the constraint (4.38) in Sec. IV B on setting  $\varphi_e = 0$  there. For the vector  $z_{12}$  between the ends of the needle and for the distance  $r_{I,y} \equiv \text{Im} z_I \equiv \text{Im}(z_1 + z_2)/2$  of the needle center from the boundary, one obtains

$$\begin{aligned} z_{12} &\rightarrow 4|A|(1 + 2h^2 \cos^2 \varphi_1) e^{-i\{\varphi_1 + [-\pi + g(\varphi_1; h)]/2\}} + \mathcal{O}(h^3), \\ r_{I,y} &\rightarrow \frac{|A|}{2h} \{1 + 2h^2[1 + 2\cos(2\varphi_1)] + \mathcal{O}(h^3)\}. \end{aligned} \quad (4.50)$$

The direction of the needle, given by the phase factor of  $z_{12}$  in Eq. (4.50), is consistent with the general expression on the right hand side of Eq. (4.48).

## 2. Force and torque

The force and the torque which the boundary exerts on needle 12 are again given by Eqs. (2.3)-(2.5) and (2.7)-(2.11) together with (4.22)-(4.24), except that now  $dz/dw$  and  $S(w)$  follow from (4.45). It is most convenient to use the integration path  $\mathcal{C}$  in Eqs. (2.5) and (2.11) along the inner boundary circle of the annulus. Since there is no force  $f_x$  parallel to the boundary, the imaginary part of  $\tau$  must vanish.

For a distant needle the force is determined by  $\tau^{(T)}$  for  $h \ll 1$ . In this regime Eqs. (2.5), (4.22), and (4.47) imply  $\tau^{(T)}/t(h) \rightarrow 2h/|A|$ , and using Eqs. (4.50), one obtains

$$\frac{f_y}{k_B T} \rightarrow -\frac{1}{r_{I,y}} t(h \ll 1), \quad h \rightarrow \frac{|z_{12}|}{8r_{I,y}}, \quad (4.51)$$

where  $t(h \ll 1)$  is taken from Eqs. (4.26), and (4.27). As expected, this result is in agreement with (4.44) in Sec. IV B for  $r_{I,x} \rightarrow +\infty$ . An orientation dependence of the needle only appears in higher order and is determined explicitly for the needle geometry considered here with the SPOE in Appendix B 2 c . As for  $\tau^{(S)}$ , we have checked that its leading  $h$  power is higher than  $h^3$ , so that  $\tau^{(S)}$  does not contain a term proportional to  $|z_{12}|^2/r_{I,y}^3$ . This is consistent with the vanishing of the stress-tensor average in the half plane and the absence in the SPOE of a hyper-universal contribution  $\propto |z_{12}|^2$  to the free energy, force, and torque.

The leading contribution to the torque  $\Theta$  is determined by  $\theta^{(T)}$  and is given by

$$\frac{\Theta}{k_B T} \rightarrow -\text{Re} \theta^{(T)} \rightarrow 6h^2 \sin(2\varphi_1) t(h \ll 1); \quad \sin(2\varphi_1) \rightarrow \sin(2\Phi_{12}), \quad h \rightarrow \frac{|z_{12}|}{8r_{I,y}}, \quad (4.52)$$

where the  $h$ -expansion of  $\text{Re } \theta^{(T)}/t(h)$  is derived in Appendix B 1 b and Eqs. (4.48)-(4.50) have been used. Equations (4.51) and (4.52) are consistent with the SPOE results in Appendix B 2 c.



## V. RESULTS FOR ARBITRARY NEEDLE LENGTHS

We now consider some simple needle geometries in which the needle length is neither very large nor very small compared to the distance between the needles. Calculating the force and torque requires the full machinery described in Sec. IV for arbitrary values of the mapping parameter  $h$  in the interval  $0 < h < 1$ . Unlike the completely analytic approaches for semi-infinite needles in Sec. III and for short needles (small  $h$  expansion) in Sec. IV, we now resort to numerical evaluation, which, however, yields results over the entire range from small to large needle lengths [45]. Actually, we restrict our attention to needle configurations for which the six mapping parameters are restricted to subspaces of lower dimension. These include the configurations (A)-(D) introduced between Eqs. (4.15) and (4.16) and the configurations of a finite needle in the half plane, discussed in Sec. IV C.

(A) For the symmetric-perpendicular configuration (A) defined in Sec. IV A 1 and shown in Fig. 1, the force component  $f_y$  and the torque on needle  $I$  vanish, due to symmetry. The component  $f_x$  is attractive ( $f_x < 0$ ) for needle universality classes  $OO, ++$  and repulsive ( $f_x > 0$ ) for classes  $+-, O+$ . We consider the case of equal needle lengths  $D$ , in which  $Df_x/(k_BT)$ , apart from the universality classes, only depends on  $\tilde{c} = c/D$ , where  $c = z_I - z_4$  is the minimum distance between the needles. In Fig. 2 our exact numerical results for  $Df_x/(k_BT)$  in the region  $10^{-2} < \tilde{c} < 10^2$  are indicated by full points. For large and small  $\tilde{c}$  there is excellent agreement with the asymptotic behavior (B32), (B33) for short needles and with the results of Sec. III for semi-infinite needles, respectively. In the latter limit,  $\tilde{c} \rightarrow 0$ , and the force  $f_x$  becomes independent of  $D$  and is given by  $f_y$  in Eq. (3.6) with  $\alpha = \pi/2$  and  $r_y(1) = c$ .

(B) In the symmetric-parallel configuration (B) defined in Sec. IV A 1 and shown in Fig. 1, the force component  $f_y$  and the torque also vanish by symmetry. We have evaluated  $f_x$  numerically in two special cases, B1 and B2. In case B1, which is denoted by (ii) in Sec. IV A 1 and belongs to classes (B) and (C), both needles have the same length  $D$ . In case B2 we denote the length of needle  $I$  by  $D$  and choose the length of needle  $II$  equal to the needle separation  $c = z_I - z_{II}$ . The dependence of  $cf_x$  on  $D/c$  in both cases, B1 and B2, is shown in Fig. 3. Again our numerical results (points) merge nicely with the asymptotic expressions (curves) for small and large  $D/c$ . For small  $D/c$  these follow from Eqs. (B35) and (B36) with  $\alpha = 0$  in case B1 and from Eqs. (B58)-(B60) in case B2. For large  $D/c$

case B1 reduces to an infinitely long strip, and  $cf_x/(k_BT) \rightarrow \Delta D/c$ , where  $\Delta = \pi(\tilde{t} - 1/48)$ , with  $\tilde{t}$  given in Eq. (3.4), is the corresponding Casimir amplitude [42]. For large  $D/c$  case B2 reduces to a needle  $II$  parallel to the boundary of a half plane, a geometry considered in the last two paragraphs of this section, and  $cf_x/(k_BT)$  is the same as  $Df_y/(k_BT)$  for  $\tilde{B} = 1$  and  $\Phi = 0$  in Fig. 8 .

(C) For the mirror-symmetric configuration (C) defined in Sec. IV A 1 and shown in Fig. 1, the force component  $f_y$  vanishes for all angles  $\pi - 2\Phi_{12} \equiv \alpha$  between the needles. The component  $f_x$  and the torque  $\Theta$  are non-zero, with the exception of the torque at  $\alpha = 0$  and  $\pi$ . Figure 4 shows  $f_x$  and  $\Theta$  for needles forming an angle  $\alpha = \pi/5 = 36^\circ$  and with lengths ranging from short to long. The quantities  $cf_x/(k_BT)$  and  $\Theta/(k_BT)$  are plotted as functions of  $D/c \equiv 1/\tilde{c}$  where the minimum separation  $c = z_{24} = z_2 - z_4$  of the needles is the distance between the two lower needle ends. For small  $D/c$  the numerical data (full points) merge nicely with the results of the small needle expansion given in Eqs. (B35), (B36). For large  $D/c$  the data for the ( $D$ -independent) force are in excellent agreement with the corresponding force for mirror-symmetric semi-infinite needles following from Eqs. (3.9), (3.11) with  $b = 1$ . The torque appears to vary linearly with  $D$  for large  $D$ , in agreement with the analytic argument at the end of Appendix C.

(D) Next we consider the force and torque for the antiparallel configuration defined in Sec. IV A 1. We consider needles of equal length antiparallel and parallel to the  $x$ -axis [41], illustrated by Fig. 1-(D'), with  $z_{34} = -z_{12} = D$ . The configuration is uniquely specified by the value of  $D$ , the vertical separation of the needles  $W = r_{1,y} - r_{4,y}$  and the relative horizontal displacement  $r_{4,x} - r_{1,x}$ . For the fixed ratio  $(r_{4,x} - r_{1,x})/W = 1.4$ , Figs. 5 and 6, show our numerical results (points) for  $Wf_x/(k_BT)$ ,  $Wf_y/(k_BT)$ , and  $\Theta/(k_BT)$  as functions of  $D/W$ . For small  $D/W$  we show the small needle prediction following from Eqs. (B41)-(B46). For large  $D/W$  the perpendicular force component  $f_y$  is dominated by the usual Casimir force for a long strip [28], so that  $Wf_y/(k_BT) \rightarrow (D/W)\Delta$  with  $\Delta$  from [42]. This and the behavior of the ( $D$ -independent) parallel force component  $f_x$  and of the torque  $\Theta$  for large  $D/W$ , derived in Eqs. (C4), (C5) and (C3), respectively, are also indicated by full lines in the figures.

Finally we consider the force and torque on a single needle in the upper half plane for various ratios  $\tilde{B} \equiv r_{I,y}/D$  of the distance of the needle center from the boundary to the needle length and for various angles  $\Phi_{12} \equiv \Phi$  between the needle and the boundary. The

torque vanishes by symmetry for  $\Phi = 0$  and  $\Phi = \pm\pi/2$ . The results for  $\tilde{B} = 10$  and  $0 < \Phi < \pi/2$ , shown in Fig. 7, are in perfect agreement with the predictions (B55) and (B56) of the operator expansion for a distant needle. In this case  $f_y - f_y|_{\Phi=\pi/4}$  and  $\Theta$  are odd and even functions, respectively, of  $\Phi - \pi/4$ . Figure 8 shows corresponding results for the intermediate distance ratio  $\tilde{B} = 1$ , where the minimum distance between the needle and the boundary, which corresponds to the perpendicular orientation  $\Phi = \pi/2$ , is half the length of the needle. As expected, there are significant deviations from our operator expansion of low order, in particular for the force near  $\Phi = \pi/2$ .

For  $\tilde{B} < 1/2$  the needle touches the boundary before attaining the perpendicular orientation, and the force and torque diverge. Figure 9 shows the case  $\tilde{B} = \sqrt{3}/4 = 0.433$  for  $0 < \Phi < \pi/3$ , with diverging results as  $\Phi$  approaches the angle  $\pi/3$  and the distance  $r_{2,y} = \frac{1}{2}D(\sin \frac{\pi}{3} - \sin \Phi)$  of the needle tip  $z_2$  from the boundary shrinks to zero. Since the divergence is a local effect, for  $r_{2,y} \ll D$  one expects  $f_y$  to be independent of  $D$  and the same as the force (3.6) on a semi-infinite needle with the same endpoint  $r_{2,y} \equiv r_y(1)$  and angle  $\Phi \equiv \alpha$  in the notation of Subsec. III A. From Eq. (3.6) we obtain  $Df_y/k_BT \approx -\frac{1}{12}(5 - 216\tilde{t})\left(\frac{\pi}{3} - \Phi\right)^{-1}$  for the leading divergent term, which is plotted in Fig. 9. The exact numerical data (points) in the figure are in excellent agreement with this prediction, and for all four sets of boundary conditions it gives an astonishingly good fit over the entire range  $0 < \Phi < \pi/2$ . Heuristic arguments (see last two paragraphs of Appendix C) suggest that the torque also diverges as  $\left(\frac{\pi}{3} - \Phi\right)^{-1}$ , and the exact numerical data in Fig. 9 appear to support this.

## VI. SUMMARY AND CONCLUDING REMARKS

The Casimir interaction of particles immersed in a binary liquid mixture near a critical point of miscibility has a long range and universal character, and non-spherical particles experience both a force and a torque. We consider the interaction of two needle shaped particles right at the critical point of a two-dimensional fluid in the Ising universality class. While particular needle configurations have been considered before [29], the approach of this paper allows us to calculate the interaction for two needles of arbitrary lengths, separations, and orientations for various combinations of surface universality classes [21].

As in earlier work [12, 24, 29, 30, 44], we utilize the conformal invariance of two-

dimensional critical systems and generate the needle geometry of interest from a simpler standard geometry by means of a conformal mapping. As outlined in Sec. II and Appendix A, we work with the stress tensor, which has well-understood conformal transformation properties, is known in the simple standard geometry, and determines the force and torque in the needle geometry of interest.

In Sec. III we consider arbitrary configurations of an infinite and a semi-infinite needle and of two semi-infinite needles and obtain the results for the force given in Eqs. (3.3), (3.6) and Eqs. (3.9), (3.11), respectively. The simple form of the force follows from the simplicity of the stress tensor in the standard geometry and of the mapping generating the needles. The region outside the needles is simply connected, and the standard geometry is the upper half plane with the two needles on the  $x$  axis.

For needles of finite length the space bounded by the needles is doubly connected, and the standard geometry is an annulus with circular needles on its boundaries. The stress tensor in the annulus is known from Cardy's work [31] and summarized in Eqs. (4.22)-(4.24). The mapping onto the two-needle geometry is a special case of Akhiezer's formula [39] for mapping the annulus onto the region outside two nonoverlapping polygons, and its derivative is given by Eq. (4.2) in Sec. IV A. Two conditions (4.8) are imposed to ensure that the mapping is single valued. Searching for values of the six parameters  $h, C, \varphi_1, \dots, \varphi_4$  in Eq. (4.2) that satisfy these two conditions and generate a given needle configuration is a formidable task. The simple relation (4.14), which expresses the angle enclosed by the two needles in terms of the sum  $\varphi_1 + \dots + \varphi_4$  reduces the space of parameters in which one must search, and we have found some simple configurations (A)-(D) of the needles, discussed in Sec. IV A 1 and shown in Fig. 1, in which the space can be further reduced. In Secs. IV B and IV C we analyze the special case in which one of the needles has a finite length and the other is semi-infinite or infinite.

We have put the finite-needle approach to work in two ways.

(a) First of all we have analyzed the case of needles with separation much greater than their lengths analytically. In this regime the inner radius  $h \ll 1$  of the annulus is much smaller than the outer radius of 1, and both the small  $h$  expansion and the small-particle operator expansion (SPOE) yield information on the force and torque. Beginning in paragraph (E) of Sec. IV A 1 and continuing in Secs. IV A 2, IV B, IV C, and Appendix B, we show the consistency of these two approaches. For example, the surface-class-independent (hyperuniversal)

contribution to force and torque arising via (2.3)-(2.11) from the Schwarzian derivative of the mapping is provided within the SPOE (B15) by the stress-tensor operator (B18).

(b) Secondly, by using the same conformal mapping approach and evaluating formulas numerically, we have studied the force and torque over the full range from small to large values of the ratio of needle length to needle separation. Results for several types of needle configurations (see Fig. 1) and several combinations of universality classes are shown in Figs. 2-9 and discussed in Sec. V. In all cases the force is attractive for  $OO$  and  $++$  boundaries and repulsive for  $+-$  and  $O+$ . For small and large values of the needle length to separation ratio, the numerical results are in excellent agreement with the SPOE results and the results for needles of infinite length in Sec. III, respectively. The linear dependence of the torque on long needles on the needle length, discussed in Appendix C, is also confirmed by the data in Figs. 4 and 6.

In this paper we have concentrated on needles in two dimensions with identical surface universality classes on both sides of the needle. The approach for needles of infinite length in Sec. III can easily be extended to needles with different boundary conditions on the two sides, for example, a needle along the  $x$ -axis with its upper edge in the class  $-$  and its lower edge in the class  $+$ . This is discussed in Appendix D, and explicit results are given for the force between (i) two semi-infinite collinear needles and (ii) a semi-infinite needle perpendicular to the boundary of the half plane. In the latter case we also consider a boundary with “chemical steps” [52], which separate the  $x$  axis into segments with  $+$  and  $-$  boundary conditions, and we calculate both the normal and lateral force on the needle. Lateral Casimir forces on colloidal particles in three dimensions exposed to a chemically structured surface have been measured in Ref. 53.

We close by comparing the advantages and disadvantages of the conformal mapping approach of Secs. IV and V and of the approach based on the small-particle operator expansion (SPOE). For evaluating the force and torque for arbitrary size to separation ratios, as in Sec. V, the former is clearly superior. However, it is limited to two-dimensional critical systems with conformal symmetry, to particle surfaces with uniform boundary conditions, and to the interaction of two particles immersed in the critical medium. The SPOE is only applicable if the particle size is small compared to the interparticle separation and to the correlation length of the medium in which the particles are immersed. However, the SPOE is not limited to two dimensions, is valid in near-critical as well as critical systems, and also

applies if there are more than two immersed particles and if the particles have non-uniform boundary conditions. In addition to spherical and nonspherical particles embedded in near critical fluids [24, 44], the SPOE method has been applied to particles bound to fluctuating surfaces in Ref. [54], where it is called “effective field theory”.

## Appendix A: TRANSLATION AND ROTATION OF ONE OF TWO PARTICLES

A general infinitesimal coordinate transformation

$$\hat{\mathbf{r}} = \mathbf{r} + \mathbf{a}(\mathbf{r}) \quad (\text{A1})$$

changes the geometry of a critical system, including the sizes, shapes, separations, and orientations of embedded particles, from  $G$  to  $\hat{G}$ . The corresponding change in the universal scaling part [46] of the free energy is given by

$$F_{\hat{G}} - F_G = -k_B T_c \int d\mathbf{r} \sum_{k,\ell} [\partial a_k(\mathbf{r}) / \partial r_\ell] \langle T_{k\ell}(\mathbf{r}) \rangle_G \quad (\text{A2})$$

to first order in  $\mathbf{a}$ , where  $T_{k\ell}$  is the stress tensor [28, 47].

For two particles in the  $(r_x, r_y)$  plane, the force and torque on particle  $I$  due to particle  $II$  follow directly from the change in free energy as particle  $I$  is translated by an infinitesimal vector  $(dR_x, dR_y)$  or rotated by an infinitesimal angle  $d\Phi$  about a point  $(r_{0,x}, r_{0,y})$ , while keeping particle  $II$  fixed. Assuming that particles  $I$  and  $II$  are located above and below the line  $r_y = \tilde{r}_y$ , respectively, we fix particle  $II$  by choosing

$$(a_x(\mathbf{r}), a_y(\mathbf{r})) = (A_x(\mathbf{r}), A_y(\mathbf{r})) \times \Theta(r_y - \tilde{r}_y), \quad (\text{A3})$$

where  $\Theta$  is the standard unit step function. To translate and rotate  $I$ , we choose

$$(A_x, A_y) = (dR_x, dR_y), \quad (\text{A4})$$

$$(A_x(\mathbf{r}), A_y(\mathbf{r})) = (-r_y + r_{0,y}, r_x - r_{0,x}) d\Phi, \quad (\text{A5})$$

respectively. On substituting

$$\partial a_k / \partial r_\ell = [\partial A_k / \partial r_\ell] \times \Theta(r_y - \tilde{r}_y) + A_k \delta_{\ell,y} \delta(r_y - \tilde{r}_y) \quad (\text{A6})$$

in Eq. (A2), the first term on the right hand side does not contribute, since  $\partial A_k / \partial r_\ell$  vanishes for the shift and is antisymmetric in  $k\ell$  for the rotation while  $T_{k\ell}$  is symmetric. Thus,

$$F_{\hat{G}} - F_G = -k_B T_c \int_{-\infty}^{\infty} dr_x J, \quad J = \sum_{k=x,y} A_k(r_x, \tilde{r}_y) \langle T_{ky}(r_x, \tilde{r}_y) \rangle_G. \quad (\text{A7})$$

Of course,  $F_{\hat{G}} - F_G$  should not depend on the precise choice of  $\tilde{r}_y$ , and this property follows from the vanishing of  $\partial(\int_{-\infty}^{\infty} dr_x J) / \partial \tilde{r}_y$  due to the continuity equation  $\sum_\ell \partial \langle T_{k\ell}(\mathbf{r}) \rangle_G / \partial r_\ell = 0$

at any point  $\mathbf{r}$  outside the particles. Using the relations

$$\begin{aligned}\langle T_{yy}(r_x, \tilde{r}_y) \rangle_G &= -\langle T_{xx}(r_x, \tilde{r}_y) \rangle_G = \text{Re } \vartheta(z), \quad \langle T_{xy}(r_x, \tilde{r}_y) \rangle_G = \langle T_{yx}(r_x, \tilde{r}_y) \rangle_G = \text{Im } \vartheta(z), \\ \vartheta(z) &\equiv \langle T(z) \rangle_G / \pi, \quad z = r_x + i\tilde{r}_y\end{aligned}\tag{A8}$$

between the Cartesian components and complex form the of stress tensor (see [28, 47]), one finds

$$J = dR_x \text{Im } \vartheta(z) + dR_y \text{Re } \vartheta(z),\tag{A9}$$

$$J = d\Phi \text{Re}[(z - z_0)\vartheta(z)], \quad z_0 = r_{0x} + ir_{0y},\tag{A10}$$

for the translation and rotation, respectively. Together with Eq. (A7) and  $dr_x = dz$ , this implies

$$F_{\hat{G}} - F_G = k_B T \left[ dR_x \text{Im} \int_{\mathcal{C}_I} dz \vartheta(z) + dR_y \text{Re} \int_{\mathcal{C}_I} dz \vartheta(z) \right],\tag{A11}$$

$$F_{\hat{G}} - F_G = k_B T d\Phi \text{Re} \int_{\mathcal{C}_I} dz (z - z_0) \vartheta(z),\tag{A12}$$

where the closed integration contour  $\mathcal{C}_I$  goes clockwise around particle  $I$ , with particle  $II$  outside the contour. In arriving at this result, we first deformed the integration path in Eq. (A7) to a counterclockwise loop around needle  $I$ , as allowed by the analyticity [28] and large  $z$  properties of  $\vartheta(z)$  and of  $(z - z_0)\vartheta(z)$ . We then replaced this integral by minus the integral around the clockwise contour  $\mathcal{C}_I$ .

Equations (A11) and (A12) are more general than our derivation and also apply to configurations in which the two particles do not lie above and below a line parallel to the  $r_x$  axis. The same is true of the corresponding expressions (2.4) and (2.8) for the force and the torque. Two needles can always be separated by a straight line, and after an appropriate global rotation of the system, Eq. (A3) can be applied. On rotating counterclockwise by an arbitrary finite angle  $\omega$ ,  $\langle T(z) \rangle \rightarrow e^{-2i\omega} \langle T(z) \rangle$ ,  $dz \rightarrow e^{i\omega} dz$ ,  $z - z_I \rightarrow e^{i\omega}(z - z_I)$ , and Eqs. (2.4) and (2.8) correctly predict the rotation  $f_x + if_y \rightarrow e^{i\omega}(f_x + if_y)$  of the force and that the torque is unchanged. Expressions (A11) and (A12) hold for two particles of arbitrary shape, even if they are positioned so that no separating straight line exists. This follows from a modified infinitesimal transformation (A3) in which the region onto which the step function  $\Theta$  projects is not a half plane.

For the special case of two widely separated needles we have checked the consistency of Eqs. (A11) and (A12) with the small-particle operator expansion reviewed in Appendix B 2.



## Appendix B: EXPANSIONS FOR SHORT NEEDLES

The Casimir interaction of a needle which is short compared to the distance to other particles and to the boundary can be studied analytically in a power series expansion. In Appendix B 1 we consider the small  $h$  expansion, where  $h$  is the ratio of the inner to outer radius of the annulus, and provide more details on the derivation of the distant needle results for force and torque presented in Sec. IV. In Appendix B 2 we study the interaction of the needles with the small-particle operator expansion (SPOE). Since the two methods must lead to identical results, one can make useful checks.

### 1. Expanding for small $h$

#### *a. Two small needles*

To arrive at the form of  $\tau^{(S)}$  for widely separated needles given in (4.25), we expand the Schwarzian derivative on the circle  $\mathcal{C} = \mathcal{C}_c$ , defined below Eq. (4.24), in  $h$ , obtaining

$$S(w = h^{1/2}Ce^{i\varphi}) \times C^2e^{2i\varphi}/6 =: \sigma(\varphi) = \sigma_0(\varphi) + h^{1/2}\sigma_1(\varphi) + h\sigma_2(\varphi) + \mathcal{O}(h^{3/2}), \quad (\text{B1})$$

where

$$\begin{aligned} \sigma_0 &= -C^2e^{2i\varphi}e^{-2i\varphi_1} - C^{-2}e^{-2i\varphi}e^{2i\varphi_3}, \\ \sigma_1 &= 2\{C^3e^{2i\varphi}(e^{-i\varphi_1} - e^{-3i\varphi_1}) + C^{-3}e^{-2i\varphi}(e^{i\varphi_3} - e^{3i\varphi_3})\}, \\ \sigma_2 &= C^4e^{2i\varphi}[-1 + e^{-2i\varphi_1}(6 - 4e^{i\varphi}) + e^{-4i\varphi_1}(-5 + 4e^{i\varphi} - 2e^{2i\varphi})] \\ &\quad + 2e^{-2i\varphi_1}e^{2i\varphi_3} \\ &\quad + C^{-4}e^{-2i\varphi}[-1 + e^{2i\varphi_3}(6 - 4e^{-i\varphi}) + e^{4i\varphi_3}(-5 + 4e^{-i\varphi} - 2e^{-2i\varphi})]. \end{aligned} \quad (\text{B2})$$

The invariance of the right hand sides on exchanging  $(C, \varphi, \varphi_1, \varphi_3) \leftrightarrow (C^{-1}, -\varphi, -\varphi_3, -\varphi_1)$  presumably persists in higher order. On the circle  $\mathcal{C}_c$  the prefactor of the square bracket in the integral (2.5) reads

$$\frac{1}{z'(w)} = -\frac{1}{A}h^2(1 - e^{i\varphi})^2e^{2i\varphi_3}[1 + h^{1/2}\delta_1(\varphi) + h\delta_2(\varphi) + \mathcal{O}(h^{3/2})], \quad (\text{B3})$$

where

$$\begin{aligned} \delta_1 &= 2C^{-1}(e^{i\varphi_3} - e^{-i\varphi_3}), \\ \delta_2 &= -4C^{-2} + C^{-2}e^{2i\varphi_3}(4 - 2e^{-i\varphi} + e^{-2i\varphi}) + C^2e^{-2i\varphi_1}(-2e^{i\varphi} + e^{2i\varphi}), \end{aligned} \quad (\text{B4})$$

and implies

$$\tau^{(S)} \equiv \int_{\mathcal{C}_c} dw \frac{1}{dz/dw} (-) \frac{1}{24} S(w)/\pi = \frac{ih^{5/2}}{4\pi AC} e^{2i\varphi_3} I, \quad (\text{B5})$$

where

$$\begin{aligned} I &= \int_0^{2\pi} d\varphi (e^{-i\varphi} - 2 + e^{i\varphi}) (1 + h^{1/2}\delta_1 + h\delta_2 + \dots) (\sigma_0 + h^{1/2}\sigma_1 + h\sigma_2 + \dots) \\ &= h \int_0^{2\pi} d\varphi (e^{-i\varphi} - 2 + e^{i\varphi}) (\sigma_2 + \delta_2\sigma_0) + \mathcal{O}(h^{3/2}) \\ &= 8\pi h e^{-2i\varphi_1} e^{2i\varphi_3}. \end{aligned} \quad (\text{B6})$$

Together with Eq. (4.21) this leads to the result for  $\tau^{(S)}$  in Eq. (4.25).

Next we derive (4.30) of  $\theta^{(S)}$  for two widely separated needles. Since in (2.11) we again integrate  $w$  counterclockwise around the circle  $\mathcal{C} = \mathcal{C}_c$ , the two required quantities  $\zeta_n(w)$  are conveniently obtained by splitting the  $\tilde{w}$  integration paths in Eq. (2.9) into three parts:  $[\alpha]$  from  $w_1$  or  $w_2$  along the outer boundary circle to the point  $-1$ ;  $[\beta]$  from  $-1$  along the negative real axis to  $-Ch^{1/2}$ ;  $[\gamma]$  from  $-Ch^{1/2}$  along the circle  $\mathcal{C}_c$  to the point  $w \equiv Ch^{1/2} \exp(i\varphi)$ . This yields

$$\zeta_n(w) = \zeta_n^{[\alpha]} + \zeta^{[\beta]} + \zeta^{[\gamma]}(w), \quad n = 1, 2 \quad (\text{B7})$$

where

$$\begin{aligned} \zeta_n^{[\alpha]}/A &= \mathcal{O}(1/h), \quad \zeta^{[\beta]}/A = -h^{-3/2}(C/2)e^{-2i\varphi_3} + \mathcal{O}(1/h), \\ \zeta^{[\gamma]}(w)/A &= -h^{-3/2}C e^{-2i\varphi_3} \left\{ \left[ 1 - h^{1/2}2C^{-1}(e^{i\varphi_3} - e^{-i\varphi_3}) \right] \left( \frac{1}{1-\Omega} - \frac{1}{2} \right) \right. \\ &\quad \left. + h \left[ 4C^{-2}(-1 + e^{-2i\varphi_3}) \left( \frac{1}{1-\Omega} - \frac{1}{2} \right) + C^{-2}e^{2i\varphi_3} \left( \frac{1}{1-\Omega} + \frac{1}{\Omega} + \frac{1}{2} \right) \right. \right. \\ &\quad \left. \left. + C^2 e^{-2i\varphi_1} \left( \frac{1}{1-\Omega} - \Omega - \frac{3}{2} \right) \right] + \mathcal{O}(h^{3/2}) \right\}, \end{aligned} \quad (\text{B8})$$

and  $\Omega \equiv e^{i\varphi}$ . By construction, only the first term  $\zeta_n^{[\alpha]}$  in Eq. (B7) depends on  $n$ , and only the third term  $\zeta^{[\gamma]}(w)$  depends on  $w$ . The  $h$ -expansion for  $\theta^{(S)}$  is obtained by substituting  $1/z'(w)$ ,  $S(w)$  and  $\zeta_n$  from Eqs. (B1)-(B4) and (B7), (B8) in Eq. (2.11). One (readily) finds that  $\zeta_n^{[\alpha]}$  and (with more work) that  $\zeta^{[\gamma]}(w)$  only contribute to  $\theta^{(S)}$  in orders higher than  $h^2$ , while  $\zeta^{[\beta]}$  makes the leading contribution  $\theta^{(S)} \rightarrow \zeta^{[\beta]}\tau^{(S)}$  given in Eq. (4.30), which is of order  $h^2$ . Here Eqs. (B5) and (B6) have been used in the last step.

*b. A small needle in the half plane*

Here we derive, within the  $h$ -expansion, the contribution  $-\text{Re } \theta^{(T)}$  to the torque acting on a small needle in the half plane shown in Eq. (4.52). For the integration path  $\mathcal{C}$  in Eq (2.11), we use the inner boundary circle  $w = he^{i\varphi}$  and split the integrations for  $\zeta_n(w)$ , as in Eq. (B7), where for  $[\alpha]$  the integration is as before, while for  $[\beta]$  and  $[\gamma]$  it goes from  $-1$  to  $-h$  and from  $-h$  to  $w = he^{i\varphi}$ , respectively. For  $[\gamma]$  we integrate over the segment of the inner circle which does not contain the singular point  $w = h$  of  $z'(w)$ . Instead of  $\varphi$  it is convenient to use the deviation  $\chi = \varphi - \pi$  from  $\varphi = \pi$  as the integration variable on the inner circle, and with the help of Eq. (4.47), one obtains

$$\begin{aligned} \zeta^{[\gamma]}(w = he^{i\varphi})/|A| &\equiv -\frac{1}{h} \int_0^\chi d\chi' \frac{1}{4 \cos^2(\chi'/2)} \tilde{\mathcal{P}}(\chi') \\ &\rightarrow -\frac{1}{2h} \tan(\chi/2) + h \left[ 2(1 - \cos \chi) \sin(2\varphi_1) + (2 \sin \chi - \tan(\chi/2)) \cos(2\varphi_1) \right]. \end{aligned} \quad (\text{B9})$$

Here  $\tilde{\mathcal{P}}$  is the product in Eq. (4.47), and we have used its behavior for small  $h$ ,

$$\begin{aligned} \tilde{\mathcal{P}}(\chi) &\equiv \prod_{k=1}^{\infty} \frac{\prod_{n=1,2} |1 + h^{2k-1} e^{i(\chi - \varphi_n)}|^2}{|1 + h^{2k} e^{i\chi}|^4} \\ &\rightarrow 1 - 2h^2 \{ [2 \sin \chi + \sin(2\chi)] \sin(2\varphi_1) + [2 \cos \chi + \cos(2\chi)] \cos(2\varphi_1) \}. \end{aligned} \quad (\text{B10})$$

To first order in  $h$ ,

$$(\zeta_1^{[\alpha]} + \zeta_2^{[\alpha]})/(2|A|) = -i(1 + e^{-2i\varphi_1}) - 4he^{-2i\varphi_1} \sin \varphi_1, \quad (\text{B11})$$

$$\left( \frac{\zeta_1^{[\alpha]} + \zeta_2^{[\alpha]}}{2} + \zeta^{[\beta]} \right)/|A| = -\frac{i}{2h} + ih \left( -1 + \frac{1}{2} e^{2i\varphi_1} - \frac{5}{2} e^{-2i\varphi_1} \right), \quad (\text{B12})$$

so that

$$\text{Re} \left( \frac{\zeta_1^{[\alpha]} + \zeta_2^{[\alpha]}}{2} + \zeta^{[\beta]} \right)/|A| = -3h \sin(2\varphi_1), \quad (\text{B13})$$

and Eq. (4.52) then follows from

$$\begin{aligned} \frac{2\pi}{t(h)} \text{Re } \theta^{(T)} &\equiv h \int_{-\pi}^{\pi} d\chi \, 4 \cos^2(\chi/2) \frac{1}{\tilde{\mathcal{P}}(\chi)} \times \\ &\quad \times \left\{ \text{Re} \left( \frac{\zeta_1^{[\alpha]} + \zeta_2^{[\alpha]}}{2} + \zeta^{[\beta]} \right) + \zeta^{[\gamma]}(he^{i\varphi}) \right\} / |A|. \end{aligned} \quad (\text{B14})$$

Inserting Eq. (B9) in Eq. (B14), one finds that  $\zeta^{[\gamma]}$  does not contribute to the leading order result shown on the right hand side of Eq. (4.52).

## 2. Operator expansion for a distant needle

Like a product of two operators in the “operator-product expansion” [34], a small particle can be represented by a sum of operators with appropriate prefactors [24, 44], see also [12]. Consider a distant needle  $J$ , i.e., a needle of short [33] length  $D_J$  and surface universality class  $H_J$ , with center at  $\mathbf{r}_J$  and directed along the unit vector  $\mathbf{n}_J$ . Inserting it into the  $d = 2$  Ising model at the critical point changes the Boltzmann weight of the corresponding field theory by a factor

$$e^{-\delta\mathcal{H}_J} \propto 1 + s_J, \quad (\text{B15})$$

where  $s_J$  is the operator series [29]

$$\begin{aligned} s_J = \sum_{\mathcal{O}=\phi,\epsilon} \mathcal{A}_{\mathcal{O}}^{(H_J)} \left( \frac{D_J}{2} \right)^{x_{\mathcal{O}}} & \left\{ 1 + \left( \frac{D_J}{2} \right)^2 \left[ \frac{1}{16x_{\mathcal{O}}} \Delta_{\mathbf{r}_J} + \right. \right. \\ & \left. \left. + \frac{3}{8(1+x_{\mathcal{O}})} \left( \mathcal{D}_J - \frac{1}{2} \Delta_{\mathbf{r}_J} \right) \right] \right\} \mathcal{O}(\mathbf{r}_J) - \frac{\pi}{2} \left( \frac{D_J}{2} \right)^2 \tilde{T}(J) + \dots \end{aligned} \quad (\text{B16})$$

Here  $\Delta_{\mathbf{r}}$  is the Laplacian operator, and the expressions

$$\mathcal{D}_J = \sum_{k,\ell=x,y} n_{J,k} n_{J,\ell} \partial_{r_{J,k}} \partial_{r_{J,\ell}} \quad (\text{B17})$$

and

$$\tilde{T}(J) = \sum_{k,\ell=x,y} n_{J,k} n_{J,\ell} T_{k\ell}(\mathbf{r}_J) \quad (\text{B18})$$

are the second derivative and the component of the stress tensor [47], respectively, in the needle direction. In Eq. (B16) all the operators  $\mathcal{O}$  are subtracted so that their bulk mean values vanish at the critical point, and  $\langle s_J \rangle_{\text{bulk}} = 0$ . The operators  $\mathcal{O} = \phi$  and  $\mathcal{O} = \epsilon$  correspond to the order parameter and energy densities, respectively, and are normalized according to

$$\langle \mathcal{O}(\mathbf{r}) \mathcal{O}(\mathbf{r}') \rangle_{\text{bulk}} = |\mathbf{r} - \mathbf{r}'|^{-2x_{\mathcal{O}}}, \quad (\text{B19})$$

with  $x_{\phi} = 1/8$  and  $x_{\epsilon} = 1$ . The universal quantities  $\mathcal{A}_{\mathcal{O}}^{(H_J)}$  in (B16) are the amplitudes of the corresponding density profiles  $\langle \mathcal{O}(r_x, r_y) \rangle_{\text{uhp}} = \mathcal{A}_{\mathcal{O}}^{(H_J)} r_y^{-x_{\mathcal{O}}}$  in the upper half plane (uhp) with the boundary at  $r_y = 0$  belonging to the surface class  $H_J$ . They are given by [28]

$$\mathcal{A}_{\phi}^{(O)} = 0, \mathcal{A}_{\phi}^{(+)} = -\mathcal{A}_{\phi}^{(-)} = 2^{1/8}, \mathcal{A}_{\epsilon}^{(O)} = -\mathcal{A}_{\epsilon}^{(+)} = -\mathcal{A}_{\epsilon}^{(-)} = 1/2. \quad (\text{B20})$$

The amplitudes  $\mathcal{A}_O^{(H)}$  should not be confused with the prefactor  $\mathcal{A}$  of the conformal transformation in Sec. III A. Denoting the angle between the unit vector  $\mathbf{n}_J = (n_{J,x}, n_{J,y})$  and the  $x$  axis by  $\Phi_J$  and using complex notation,

$$r_x + ir_y = z, \quad r_x - ir_y = \bar{z}, \quad n_x + in_y = e^{i\Phi}, \quad (\text{B21})$$

one obtains the useful relation

$$\tilde{T}(J) = \cos(2\Phi_J)T_{xx}(\mathbf{r}_J) + \sin(2\Phi_J)T_{xy}(\mathbf{r}_J) = -\frac{1}{2\pi} \left[ e^{2i\Phi_J}T(z_J) + e^{-2i\Phi_J}\bar{T}(\bar{z}_J) \right]. \quad (\text{B22})$$

Here  $T(z)$  and  $\bar{T}(\bar{z})$  are components of the complex stress tensor [28], and Ref. [47] was used in the last step. Note that the prefactor of the  $\tilde{T}(J)$ -term in Eq. (B16) is independent of the surface universality class  $H_J$  of the needle, i.e., “hyper-universal” [48]. The ellipsis in Eq. (B16) represents contributions from higher descendants of 1,  $\phi$ ,  $\epsilon$ , each of which is compatible with all symmetries of the needle and which, due to their scaling dimensions, are multiplied by powers of  $D_J$ , greater by at least 2 than the powers shown.

#### *a. Two small needles*

For two small needles  $I$  and  $II$  the free energy of interaction  $\delta F$  is determined by [35]

$$e^{-\delta F/(k_B T)} = 1 + \langle s_I s_{II} \rangle_{\text{bulk}}, \quad (\text{B23})$$

where, on using (B16)-(B20),

$$\langle s_I s_{II} \rangle_{\text{bulk}} = \pm \mathcal{E} + \mathcal{H} + \dots \quad (\text{B24})$$

for needle classes  $OO$  (upper sign) and  $O+$  (lower sign), while

$$\langle s_I s_{II} \rangle_{\text{bulk}} = \pm \mathcal{F} + \mathcal{E} + \mathcal{H} + \dots \quad (\text{B25})$$

for classes  $++$  (upper sign) and  $+-$  (lower sign). Here

$$\begin{aligned} \mathcal{E} &= \frac{D_I D_{II}}{16} \left[ 1 + \left( \frac{D_I}{8} \right)^2 \left( -\frac{1}{2} \Delta_{\mathbf{r}_I} + 3\mathcal{D}_I \right) + \left( \frac{D_{II}}{8} \right)^2 \left( -\frac{1}{2} \Delta_{\mathbf{r}_{II}} + 3\mathcal{D}_{II} \right) \right] \frac{1}{|\mathbf{r}_I - \mathbf{r}_{II}|^2} \\ &= \frac{D_I D_{II}}{16 |\mathbf{r}_I - \mathbf{r}_{II}|^2} \left[ 1 + 2^{-3} (\beta_I + \beta_{II}) \right], \end{aligned} \quad (\text{B26})$$

$$\begin{aligned}
\mathcal{F} &= (D_I D_{II})^{1/8} \left[ 1 + \frac{D_I^2}{12} (\Delta_{\mathbf{r}_I} + \mathcal{D}_I) + \frac{D_{II}^2}{12} (\Delta_{\mathbf{r}_{II}} + \mathcal{D}_{II}) \right] \frac{1}{|\mathbf{r}_I - \mathbf{r}_{II}|^{1/4}} \\
&= \left( \frac{D_I D_{II}}{|\mathbf{r}_I - \mathbf{r}_{II}|^2} \right)^{1/8} \left[ 1 + 2^{-6} (\beta_I + \beta_{II}) \right],
\end{aligned} \tag{B27}$$

and the hyper-universal contribution (see Refs. [28, 47]) is

$$\begin{aligned}
\mathcal{H} &= \left( \frac{\pi}{2} \right)^2 \left( \frac{D_I D_{II}}{4} \right)^2 \langle \tilde{T}(I) \tilde{T}(II) \rangle_{\text{bulk}} \\
&= 2^{-10} (D_I D_{II})^2 \left[ \frac{e^{2i(\Phi_I + \Phi_{II})}}{(z_I - z_{II})^4} + \text{c.c.} \right].
\end{aligned} \tag{B28}$$

The quantity  $\beta_J$  is defined by

$$\beta_J = \frac{D_J^2}{|\mathbf{r}_I - \mathbf{r}_{II}|^2} \left\{ -1 + 3 \left[ \left( \mathbf{n}_J (\mathbf{r}_I - \mathbf{r}_{II}) \right) / |\mathbf{r}_I - \mathbf{r}_{II}| \right]^2 \right\}, \tag{B29}$$

where the curly bracket depends on the angle between the direction of needle  $J$  and the vector between the two needle centers. As expected, all the terms in the free energy remain unchanged if either needle is rotated about its center by 180 degrees.

The force  $(f_x, f_y)$  on needle  $I$  follows from

$$(f_x, f_y) = - \left( \frac{\partial}{\partial r_{I,x}}, \frac{\partial}{\partial r_{I,y}} \right) \delta F, \tag{B30}$$

and the torque  $\Theta$  from Eq. (2.7) with  $\Phi_{12} \equiv \Phi_I$ .

For illustration, consider the symmetric perpendicular (letter T) needle configuration with needle centers on the  $x$  axis and  $r_{I,x} - r_{II,x} > 0$ , as described in paragraph (A) of section IV A 1, and assume that the two needles have equal lengths  $D_I = D_{II} \equiv D$ . Denoting by  $B = |\mathbf{r}_I - \mathbf{r}_{II}|/D \equiv (r_{I,x} - r_{II,x})/D \equiv |z_{I,II}|/D$  the center-to-center distance of the needles in units of  $D$ , one finds

$$\mathcal{E} = 2^{-4} B^{-2} + 2^{-7} B^{-4}, \quad \mathcal{F} = B^{-1/4} + 2^{-6} B^{-9/4}, \quad \mathcal{H} = -2^{-9} B^{-4}. \tag{B31}$$

The component  $f_y$  of the force on needle  $I$  vanishes, and

$$Df_x/(k_B T) = (d/dB) \ln \left[ 1 + (+1, -1) \times 2^{-4} B^{-2} + (3, -5) \times 2^{-9} B^{-4} \right] \tag{B32}$$

for needle classes  $OO$  (left entry),  $O+$  (right entry), and, via [49],

$$Df_x/(k_B T) = (d/dB) \ln \left[ 1 \pm \left( B^{-1/4} + 2^{-6} B^{-9/4} \right) + 2^{-4} B^{-2} + \mathcal{O}(B^{-4}) \right] \tag{B33}$$

for classes  $++$  (upper sign) and  $+-$  (lower sign).

As another example, consider needle configurations mirror symmetric about the imaginary axis, which correspond to class (C) in section IV A 1. By symmetry  $f_y = 0$ . In terms of the angle  $\alpha = \Phi_{34} - \Phi_{12} \equiv \Phi_{II} - \Phi_I$  enclosed by the two needles, Eqs. (B26)-(B29) lead to

$$\begin{aligned}\mathcal{E} &= 2^{-4}B^{-2} + 2^{-6}\{-1 + 3[\sin(\alpha/2)]^2\}B^{-4}, \\ \mathcal{F} &= B^{-1/4} + 2^{-5}\{-1 + 3[\sin(\alpha/2)]^2\}B^{-9/4}, \quad \mathcal{H} = +2^{-9}B^{-4}\end{aligned}\quad (\text{B34})$$

and

$$\begin{aligned}Df_x/(k_B T) &= (\partial/\partial B) \ln \left[ 1 + (1, -1) \times 2^{-4}B^{-2} + \right. \\ &\quad \left. + \{(-7, 9) + (24, -24)[\sin(\alpha/2)]^2\} \times 2^{-9}B^{-4} \right]\end{aligned}\quad (\text{B35})$$

for  $(OO, O+)$  and [49]

$$\begin{aligned}Df_x/(k_B T) &= (\partial/\partial B) \ln \left[ 1 \pm \left( B^{-1/4} + 2^{-5}\{-1 + 3[\sin(\alpha/2)]^2\}B^{-9/4} \right) + \right. \\ &\quad \left. + 2^{-4}B^{-2} + \mathcal{O}(B^{-4}) \right]\end{aligned}\quad (\text{B36})$$

for  $++$  (upper sign) and  $+-$  (lower sign). The special cases (i) and (ii) of collinear and symmetric parallel needles correspond to  $\alpha = \pi$  and  $\alpha = 0$ , respectively. For  $0 < \alpha < \pi$  needle  $II$  exerts a nonvanishing torque  $\Theta$  on needle  $I$ , where  $\Theta/(k_B T) = -(\partial/\partial \Phi_I)\delta F/(k_B T)$  is given by the right hand sides of Eqs. (B35) and (B36) with  $\partial/\partial B$  replaced by  $-\partial/\partial \alpha$ . For  $(OO, O+)$  one finds from Eq. (B35) that  $\Theta/(k_B T) = (-1, 1)2^{-7}3B^{-4}\sin \alpha + \mathcal{O}(B^{-6})$ . The sign of  $\Theta$  indicates that the interaction is dominated by the two closer needle halves.

We also consider case (D) in Sec. IV A 1, in which the two needles of equal length  $D$  form angles  $\Phi_{12} \equiv \Phi_I \equiv \Phi$  and  $\Phi_{34} = \Phi + \pi$  with the vector  $z_I - z_{II} > 0$  between their centers on the  $x$  axis. For this geometry Eqs. (B23)-(B30) yield

$$\begin{aligned}Df_x/(k_B T) &= (\partial/\partial B) \ln(1 + S), \\ S &= \pm \left( 2^{-4}B^{-2} + \{-1 + 3(\cos \Phi)^2\}2^{-6}B^{-4} \right) + \cos(4\Phi)2^{-9}B^{-4}\end{aligned}\quad (\text{B37})$$

and

$$Df_y/(k_B T) = \left( \pm 6 \sin(2\Phi) + \sin(4\Phi) \right) 2^{-7}B^{-5}/(1 + S) \quad (\text{B38})$$

for needle classes  $OO$  (upper sign) and  $O+$  (lower sign). For needle classes  $++$  (upper sign) and  $+-$  (lower sign) the force components are

$$\begin{aligned} Df_x/(k_B T) &= (\partial/\partial B) \ln(1 + S'), \\ S' &= \pm \left( B^{-1/4} + \{-1 + 3(\cos \Phi)^2\} 2^{-5} B^{-9/4} \right) + 2^{-4} B^{-2} + \\ &\quad + \{-1 + 3(\cos \Phi)^2\} 2^{-6} B^{-4} + \cos(4\Phi) 2^{-9} B^{-4} \end{aligned} \quad (\text{B39})$$

and

$$\begin{aligned} Df_y/(k_B T) &= \left[ \pm 3(\sin(2\Phi)) 2^{-5} B^{-13/4} + \right. \\ &\quad \left. + (6 \sin(2\Phi) + \sin(4\Phi)) 2^{-7} B^{-5} \right] / (1 + S'). \end{aligned} \quad (\text{B40})$$

In Section V we found it convenient to rotate this same configuration by an angle  $\pi - \Phi$ , so that needles  $I$  and  $II$  are antiparallel and parallel to the real axis and  $z_I - z_{II} = |\mathbf{r}_I - \mathbf{r}_{II}| e^{i(\pi - \Phi)}$ , implying  $[r_{I,x} - r_{II,x}, r_{I,y} - r_{II,y}] = |\mathbf{r}_I - \mathbf{r}_{II}| \times [\sin(\Phi - (\pi/2)), \cos(\Phi - (\pi/2))]$ . For this orientation

$$\begin{aligned} Df_x/(k_B T) &= \left[ \pm \left( (2B)^{-3} + (2B)^{-5} [-5 + 9(\cos \Phi)^2] \right) \cos \Phi + \right. \\ &\quad \left. + 2^{-7} B^{-5} \cos(5\Phi) \right] / (1 + S), \end{aligned} \quad (\text{B41})$$

$$\begin{aligned} Df_y/(k_B T) &= \left[ \pm \left( -(2B)^{-3} + (2B)^{-5} [2 - 9(\cos \Phi)^2] \right) \sin \Phi - \right. \\ &\quad \left. - 2^{-7} B^{-5} \sin(5\Phi) \right] / (1 + S) \end{aligned} \quad (\text{B42})$$

for needle universality classes  $OO$  (upper sign) and  $O+$  (lower sign), while for  $++$  (upper sign) and  $+-$  (lower sign)

$$\begin{aligned} Df_x/(k_B T) &= \left[ \pm \left( B^{-5/4} - 2^{-5} 3 B^{-13/4} [11 - 17(\cos \Phi)^2] \right) (\cos \Phi) / 4 + \right. \\ &\quad \left. + \left( (2B)^{-3} + (2B)^{-5} [-5 + 9(\cos \Phi)^2] \right) \cos \Phi + \right. \\ &\quad \left. + 2^{-7} B^{-5} \cos(5\Phi) \right] / (1 + S'), \end{aligned} \quad (\text{B43})$$

$$\begin{aligned} Df_y/(k_B T) &= \left[ \pm \left( -B^{-5/4} + 2^{-5} 3 B^{-13/4} [3 - 17(\cos \Phi)^2] \right) (\sin \Phi) / 4 + \right. \\ &\quad \left. + \left( -(2B)^{-3} + (2B)^{-5} [2 - 9(\cos \Phi)^2] \right) \sin \Phi - \right. \\ &\quad \left. - 2^{-7} B^{-5} \sin(5\Phi) \right] / (1 + S'). \end{aligned} \quad (\text{B44})$$



As required by symmetry,  $f_x$  and  $f_y$  in Eqs. (B37)-(B40) are even and odd in  $\Phi$ , respectively, and in Eqs. (B41)-(B44) they are odd and even in  $\Phi - (\pi/2)$ . For the torque  $\Theta$  in case (D) our operator expansion yields

$$\Theta/(k_B T) = -\left(\pm 6 \sin(2\Phi) + \sin(4\Phi)\right) 2^{-8} B^{-4} / (1 + S) \quad (\text{B45})$$

for needle classes  $OO$  (upper sign) and  $O+$  (lower sign) and

$$\begin{aligned} \Theta/(k_B T) = & -\left[\pm 3\left(\sin(2\Phi)\right) 2^{-6} B^{-9/4} + \right. \\ & \left. + \left(6 \sin(2\Phi) + \sin(4\Phi)\right) 2^{-8} B^{-4}\right] / (1 + S'). \end{aligned} \quad (\text{B46})$$

for needle classes  $++$  (upper sign) and  $+-$  (lower sign).

For two small needles with arbitrary lengths  $D_I$ ,  $D_{II}$  and angles  $\Phi_I$ ,  $\Phi_{II}$ , the SPOE reproduces the leading force contribution (4.28) and the leading hyper-universal contributions (4.29) and (4.30) to the force and torque derived from the  $h$ -expansion. For the latter quantities this is apparent from Eqs. (4.31) and (B28) since  $\delta F^{(\text{hu})}/(k_B T) = -\mathcal{H}$ .

*b. A small and a semi-infinite needle*

The interaction free energy  $\delta F$  [35] of a small needle  $I$  and a semi-infinite needle (semi), i.e., the free energy required to transfer  $I$  from the bulk plane to the plane with the semi-infinite needle, is determined by

$$e^{-\delta F/(k_B T)} = 1 + \langle s_I \rangle_{\text{semi}}. \quad (\text{B47})$$

Here  $s_I$  is the operator series in Eq. (B16), and  $\langle \rangle_{\text{semi}}$  denotes a thermal average in the  $z = r_x + ir_y$  plane with a semi-infinite needle of class  $H_{\text{semi}}$  coinciding with the positive real axis, as in Sec. IV B. Since the semi-infinite needle can be generated from the boundary of the upper half  $w$  plane by the conformal transformation  $z = w^2$ , the averages of the various operators on the right hand side of Eq. (B47) follow from their counterparts in the half plane. From Eqs. (B22) and (2.1) and the vanishing of  $\langle T(w) \rangle_{\text{half plane}}$ , we obtain

$$\begin{aligned} \langle \mathcal{O}(r_{I,x}, r_{I,y}) \rangle_{\text{semi}} &= \mathcal{A}_{\mathcal{O}}^{(H_{\text{semi}})} \left[ 2|z_I| \sin\left((\arg z_I)/2\right) \right]^{-x_{\mathcal{O}}} \\ &= \mathcal{A}_{\mathcal{O}}^{(H_{\text{semi}})} \left[ 2|z_I| \left(|z_I| - r_{I,x}\right) \right]^{-x_{\mathcal{O}}/2} \end{aligned} \quad (\text{B48})$$

and

$$\langle \tilde{T}(I) \rangle_{\text{semi}} = -\cos[2(\Phi_I - \arg z_I)] / (64\pi |z_I|^2), \quad (\text{B49})$$

where  $0 < \arg z_I < 2\pi$  and the position vector  $z_I = r_{Ix} + ir_{Iy}$  is defined below Eq. (4.36). The expression

$$\frac{f_x - if_y}{k_B T} = \frac{1}{1 + \langle s_I \rangle_{\text{semi}}} \left( \frac{\partial}{\partial r_{Ix}} - i \frac{\partial}{\partial r_{Iy}} \right) \langle s_I \rangle_{\text{semi}} \quad (\text{B50})$$

for the force, which follows from Eqs. (B47) and (B16), reproduces, in leading order, the result from the  $h$ -expansion given below Eq. (4.44). The reason is that in

$$\left( \frac{\partial}{\partial r_{Ix}} - i \frac{\partial}{\partial r_{Iy}} \right) \langle \mathcal{O} \rangle_{\text{semi}} = -x_{\mathcal{O}} \frac{3z_I - |z_I|}{2z_I(z_I - |z_I|)} \langle \mathcal{O} \rangle_{\text{semi}}, \quad (\text{B51})$$

with  $\langle \mathcal{O} \rangle_{\text{semi}}$  from Eq. (B48), the same fraction appears on the right-hand side as in Eq. (4.44), and on expressing  $\langle \mathcal{O} \rangle_{\text{semi}}$  via (4.42) in terms of  $h/|z_{12}| \equiv h/D_I$ , one may use that

$$\sum_{\mathcal{O}=\phi, \epsilon} \mathcal{A}_{\mathcal{O}}^{(H_I)} \mathcal{A}_{\mathcal{O}}^{(H_{\text{semi}})} x_{\mathcal{O}} (4h)^{x_{\mathcal{O}}} \rightarrow t(h \rightarrow 0) \equiv \{h, -h, (\sqrt{2}/8)h^{1/8}, -(\sqrt{2}/8)h^{1/8}\} \quad (\text{B52})$$

for  $\{OO, O+, ++, +- \}$ . Moreover, in the cases  $++$  and  $+-$ , the denominator on the right hand side of (B50) is consistent with the denominators in Eq. (4.27).

The orientation-dependent contribution to  $e^{-\delta F/(k_B T)}$  of lowest order in the needle length,

$$2^{-9} (D_I/|z_I|)^2 \cos[2(\Phi_I - \arg z_I)], \quad (\text{B53})$$

comes from inserting the stress tensor average (B49) in Eq. (B47), using Eq. (B16), and is independent of the needle classes  $H_I$  and  $H_{\text{semi}}$ . For universality classes  $OO$  and  $O+$ , the contribution (B53) clearly dominates the orientation dependence  $\propto D_I^{x_{\epsilon}+2} = D_I^3$  coming from the  $\mathcal{D}_I \mathcal{O}$ -term in Eq. (B16), provided that the components  $r_{Ix}$  and  $r_{Iy}$  of  $z_I$  are of the same order. However, on approaching the limit  $r_{Ix} \rightarrow +\infty$  with  $r_{Iy}$  finite, the contribution from the  $\mathcal{D}_I \mathcal{O}$ -term approaches the finite orientation dependence of a needle in the half plane (see Eq. (B54) below), while the contribution (B53) vanishes. For classes  $++$  and  $+-$  the  $\mathcal{D}_I \mathcal{O}$ -term contributes an orientation dependence preceded by a power law  $D_I^{(1/8)+2}$  with an exponent which is only slightly larger than the exponent of the power  $D_I^2$  in Eq. (B53). Note that Eq. (B53) favors needle orientations parallel and antiparallel to the vector  $z_I$  from the midpoint of the finite needle to the finite end of the semi-infinite needle.

### c. A small needle in the half plane and in the symmetric-parallel configuration

For a small needle  $I$  in the upper half plane (uhp) the free energy  $\delta F$  of interaction [35] with the boundary of surface class  $H_S$  at  $r_y = 0$  is determined by Eq. (B47), with  $\langle \rangle_{\text{semi}}$

replaced by the average  $\langle \rangle_{\text{uhp}}$  in the half plane. The expressions for  $\langle \mathcal{O} \rangle_{\text{uhp}}$  given above Eq. (B20) and the vanishing of the stress tensor average imply

$$e^{-\delta F/(k_B T)} = 1 + \sum_{\mathcal{O}=\phi, \epsilon} \mathcal{A}_{\mathcal{O}}^{(H_I)} \mathcal{A}_{\mathcal{O}}^{(H_S)} \left( \frac{D_I}{2r_{I,y}} \right)^{x_{\mathcal{O}}} \times \left[ 1 + \left( \frac{D_I}{2r_{I,y}} \right)^2 \frac{1}{16} (x_{\mathcal{O}} + 1 - 3x_{\mathcal{O}} \cos(2\Phi_I)) \right]. \quad (\text{B54})$$

Both the force and the torque follow from Eq. (B54).

The force  $f = -\partial \delta F / \partial r_{I,y}$  with  $D_I$  and  $\Phi_I$  fixed is given by

$$D_I f / (k_B T) = (\partial / \partial \tilde{B}) \ln \left\{ 1 \pm \frac{1}{4} (2\tilde{B})^{-1} \left[ 1 + (2\tilde{B})^{-2} \frac{1}{16} (2 - 3 \cos(2\Phi_I)) \right] \right\} \quad (\text{B55})$$

for classes  $OO$  (upper sign) and  $O+$  (lower sign), and by

$$D_I f / (k_B T) = (\partial / \partial \tilde{B}) \ln \left\{ 1 \pm 2^{1/4} (2\tilde{B})^{-1/8} \left[ 1 + (2\tilde{B})^{-2} \frac{3}{27} (3 - \cos(2\Phi_I)) \right] + \frac{1}{4} (2\tilde{B})^{-1} \left[ 1 + (2\tilde{B})^{-2} \frac{1}{16} (2 - 3 \cos(2\Phi_I)) \right] \right\} \quad (\text{B56})$$

for  $++$  (upper sign) and  $+-$  (lower sign). Here  $\tilde{B} = r_{I,y} / D_I$ .

The expressions for the torque per  $k_B T$ ,  $-\partial(\delta F / k_B T) / \partial \Phi_I$ , follow for the various cases of universality classes  $H_I H_S$  from the corresponding right hand sides of Eqs. (B55), (B56) on replacing  $(\partial / \partial \tilde{B})$  by  $(\partial / \partial \Phi_I)$ .

Next we consider a small needle in the symmetric-parallel configuration (B) of Fig. 1, assuming  $D_I / c \ll 1$  and  $D_{II} / c$  arbitrary, where  $c = z_I - z_{II}$  is the distance between the needles. The limits  $D_{II} / c \rightarrow \infty$  and  $D_{II} / c \ll 1$  correspond to a small needle in the half plane and configuration (B) with two small needles of different lengths, respectively. The free energy  $\delta F$  is determined by Eq. (B47) with  $\langle \rangle_{\text{semi}}$  replaced by the average  $\langle \rangle_{II}$  in the plane containing needle  $II$ . For a needle  $II$  with boundary class  $H_{II}$ , centered about the origin and extending along the  $y$  axis, the profiles of the order parameter and energy density are given by (see, e.g., Appendix A1 in the first paper of Ref. [12])

$$\langle \mathcal{O}(r_x, 0) \rangle_{II}^{(H_{II})} = \mathcal{A}_{\mathcal{O}}^{(H_{II})} (D_{II} / 2)^{-x_{\mathcal{O}}} \left[ \Xi(2|r_x| / D_{II}) \right]^{x_{\mathcal{O}}}, \quad (\text{B57})$$

$$\Xi(\xi) \equiv \xi^{-1} (\xi^2 + 1)^{-1/2}, \quad (\text{B58})$$

for  $\mathcal{O} = \phi$  and  $\mathcal{O} = \epsilon$ , respectively. Making use of this result and retaining only the leading monopole contribution in the SPOE, one obtains

$$D_{II} f_x / (k_B T) = 2(\partial / \partial \xi) \ln \left\{ 1 \pm \frac{1}{4} (D_I / D_{II}) \Xi(\xi) \right\} \Big|_{\xi=2c/D_{II}} \quad (\text{B59})$$

for classes  $OO$  (upper sign),  $O+$  (lower sign) and

$$D_{II}f_x/(k_B T) = 2(\partial/\partial\xi) \ln \left\{ 1 \pm 2^{1/4} (D_I/D_{II})^{1/8} [\Xi(\xi)]^{1/8} \right\} \Big|_{\xi=2c/D_{II}} \quad (\text{B60})$$

for  $++$  (upper sign),  $+-$  (lower sign).

## Appendix C: PARALLEL NEEDLES WITH STRONG OVERLAP AND LONG MIRROR-SYMMETRIC NEEDLES

First we consider the torque on one of the two antiparallel needles of configuration (D), introduced in Sec. IV A 1 in the limit of “strong overlap.” Two needles of equal length oriented parallel to the  $x$  axis [41] overlap strongly if the distances  $|z_1 - z_4| = |z_2 - z_3|$  between the left ends  $z_1$  and  $z_4$  and the right ends  $z_2$  and  $z_3$  of needles  $I$  and  $II$ , respectively, are much smaller than their lengths  $|z_2 - z_1| = |z_3 - z_4| = D$ , so that the two needles form boundaries of a long strip of width  $W = |r_{1,y} - r_{4,y}| = |r_{2,y} - r_{3,y}|$ . On integrating closely around needle  $I$ , which is located above needle  $II$ , in Eq. (2.8), the only contributions to the torque  $\theta$  come from regions with a width of order  $|z_1 - z_4| = |z_2 - z_3|$  near the ends of the needles, i.e., near the ends of the strip. The reason is that (i)  $\langle T(z) \rangle$  vanishes outside the strip over most of its length, i.e., over most of the upper edge of needle  $I$ , and (ii) inside the strip  $\langle T(z) \rangle$  is independent of  $z$  and equal to its value  $\pi\Delta/W^2$  in an infinite strip, with  $\Delta$  from [42]. Thus, by virtue of the odd factor  $z - z_I = r_x - r_{I,x}$  in Eq. (2.8), the interval of integration centered about  $r_{I,x} = (r_{1,x} + r_{2,x})/2$  and comprising nearly all of the lower edge of needle  $I$  gives a vanishing contribution. For large  $D/W$  the two end regions are uncorrelated, and each is equivalent to the end region of a system of two semi-infinite needles. Replacing  $z - z_I$  in the left and right end contributions by  $-D/2$  and  $D/2$ , respectively, we obtain

$$\pi\theta \rightarrow -(D/2) \lim_{d \rightarrow +\infty} \left[ \int_{\mathcal{C}_{I+}(d)} dz \langle T(z) \rangle_{\text{si}+} - \int_{\mathcal{C}_{I-}(d)} dz \langle T(z) \rangle_{\text{si}-} \right]. \quad (\text{C1})$$

Here  $\text{si}+$  denotes a system of two semi-infinite needles  $I_+$  and  $II_+$  extending from  $z_1$  and  $z_4$  to  $z_1 + |\infty|$  and  $z_4 + |\infty|$ , respectively, while  $\text{si}-$  is the system of two needles  $I_-$  and  $II_-$  extending from  $z_2$  and  $z_3$  to  $z_2 - |\infty|$  and  $z_3 - |\infty|$ . The integration path  $\mathcal{C}_{I+}(d)$  goes clockwise around the tip  $z_1$  of needle  $I_+$ , starting at  $z = z_1 + d - i0$  and ending at  $z = z_1 + d + i0$ . Similarly,  $\mathcal{C}_{I-}(d)$  goes clockwise around the tip  $z_2$  of needle  $I_-$ , starting at  $z = z_2 - d + i0$  and ending at  $z = z_2 - d - i0$ .

On rotating by 180 degrees, the  $\text{si}-$  system is mapped onto the  $\text{si}+$  system, with needle  $I_-$  mapped onto needle  $II_+$ , i.e.  $z_2$  onto  $z_4$ , and needle  $II_-$  mapped onto  $I_+$ , i.e.,  $z_3$  onto  $z_1$ . Since exchanging the universality classes in a two needle system does not change the stress tensor average [43], it is the same for the  $\text{si}+$  and rotated  $\text{si}-$  systems. Moreover, the

rotation changes  $dz \rightarrow -dz$  while no prefactor arises in front of  $T$ , and Eq. (C1) yields

$$\pi\theta \rightarrow -(D/2) \lim_{d \rightarrow +\infty} \int_{\mathcal{C}_{I+}(d) + \mathcal{C}_{II+}(d)} dz < T(z) >_{\text{si}+} . \quad (\text{C2})$$

Here the path  $\mathcal{C}_{II+}(d)$  encircles the tip  $z_4$  of needle  $II_+$  clockwise, starting at  $z_4 + d - i0$  and ending at  $z_4 + d + i0$ . The integration path in Eq. (C2) becomes connected, leading to a vanishing result, if one adds both a vertical segment from  $z_4 + d + i0$  to  $z_1 + d + r_{4,x} - r_{1,x} - i0$  and a horizontal segment from  $z_1 + d + r_{4,x} - r_{1,x} - i0$  to  $z_1 + d - i0$  to the integration path. Since for both segments  $< T(z) >_{\text{si}+}$  equals its value inside the infinite strip, and since the vertical segment leads to a purely imaginary result, the torque  $\Theta = -\text{Re } \theta$  on needle  $I$  is given by  $-1/\pi$  times the contribution of the horizontal segment, with the result

$$\Theta = -(D/2)(r_{1,x} - r_{4,x})\Delta/W^2 , \quad (\text{C3})$$

where  $\Delta$  depends on the universality classes of the two needles, as specified in Ref. [42].

We now calculate the component  $f_x$  of the force on needle  $I$  due to needle  $II$ . For this, it is convenient to place the origin at the center of reflection of the needle configuration by setting  $z_3 = -z_1$ ,  $z_4 = -z_2$  and to integrate along a path  $\mathcal{C}_I$  in Eq. (2.4) midway between the needles along the real axis from  $z = +\infty$  to  $z = -\infty$ , closing the path with a semicircle of infinite radius which does not contribute to the integral. Since  $< T(z = r_x) > = < T(z = -r_x) >$ , and since  $\text{Im} < T(z = r_x) >$  vanishes except near the ends of the needle, the desired integral over  $\text{Im} < T(z = r_x) >$  equals twice the corresponding integral with  $I$  and  $II$  replaced by their semi-infinite counterparts  $I_+$  and  $II_+$ . In this way we obtain

$$\frac{f_x}{k_B T} = \frac{\pi}{W} \left( \frac{1}{48} \frac{1 + 3b - 3b^2 - b^3}{1 + 3b + 3b^2 + b^3} - \frac{1 - b}{1 + b} \tilde{t} \right) , \quad (\text{C4})$$

where  $\tilde{t}$  is given in Eq. (3.4). Here  $b$  is positive and related by

$$(r_{1,x} - r_{4,x})/W = \frac{1}{2\pi} (2 \ln b + b - 1/b) \quad (\text{C5})$$

to the ratio  $(r_{1,x} - r_{4,x})/W$  of the parallel and perpendicular components of the vector between the two left needle ends. As expected,  $f_x/(k_B T)$  is an odd function of  $r_{1,x} - r_{4,x}$  and tends to  $\Delta/W$ ,  $-\Delta/W$ , and 0 in the cases  $b \rightarrow +\infty$ , 0, and 1 in which the ratio on the left hand side of (C5) tends to  $+\infty$ ,  $-\infty$ , and 0, respectively.

To derive Eqs. (C4) and (C5), we first generate the geometry of parallel semi-infinite needles  $I_+$  and  $II_+$  from the upper half  $w$  plane by means of the conformal transformation

[50]

$$z(w) = \frac{W}{\pi} \left[ \frac{w^2}{2b} + w \left( 1 - \frac{1}{b} \right) + \frac{1}{4} \left( \frac{1}{b} + b \right) - 1 - \ln \frac{w}{\sqrt{b}} + \frac{i\pi}{2} \right]. \quad (\text{C6})$$

Together with (C5) this transformation conveniently places the tips of  $I_+$  and  $II_+$  symmetrically about the origin, at  $z = z(1) \equiv z_1 = r_{1,x} + iW/2$  and  $z = z(-b) \equiv z_4 = -z_1$ , respectively. The integration path mentioned just above Eq. (C4), which is midway between the semi-infinite needles  $I_+$  and  $II_+$ , corresponds, according to Eqs. (2.4) and (2.5), to the imaginary axis of the upper half  $w$  plane. Similarly, the integral over  $\text{Im} < T(z = r_x) >$  corresponds to the integral of a real rational function of  $|w|$  from 0 to  $+\infty$  and leads to a force  $f_x$  on needle  $I_+$  due to  $II_+$  which is exactly half of  $f_x$  in Eq. (C4) [51].

Finally we consider the mirror-symmetric needle configuration (class (C) of Sec. IV A 1) and argue that in this case the torque  $\Theta$  also increases linearly with the needle length  $D$  for  $D \rightarrow \infty$ . First we place needles  $I$  and  $II$  so that  $z_2 = 1$ ,  $z_1 = D + 1$  and  $z_4 = e^{i\alpha} z_2$ ,  $z_3 = e^{i\alpha} z_1$ . We also introduce an auxiliary “wedge” configuration of two corresponding semi-infinite needles which extend from  $z = 0$  to  $|\infty|$  and to  $e^{i\alpha} |\infty|$ , dividing the  $z$ -plane into two wedges of opening angles  $\alpha$  and  $2\pi - \alpha$ . To evaluate  $\theta$  in (2.8), we choose  $\mathcal{C}_I$  so that it encircles needle  $I$  closely and subtract and add  $< T(z) >_{\text{wedge}}$  to  $< T(z) >$ . This leads to  $\theta = \delta\theta + \tilde{\theta}$ , where

$$\begin{aligned} \delta\theta &= \int_{\mathcal{C}_I} dz \left[ < T(z) > - < T(z) >_{\text{wedge}} \right] \left( r_x - \left( \frac{D}{2} + 1 \right) \right), \\ \tilde{\theta} &= \int_1^{D+1} dr_x < T(r_x + i0) - T(r_x - i0) >_{\text{wedge}} \left( r_x - \left( \frac{D}{2} + 1 \right) \right). \end{aligned} \quad (\text{C7})$$

Since  $< T(z) >_{\text{wedge}} \propto z^{-2}$  [28], its average in the integral for  $\tilde{\theta}$  is proportional to  $r_x^{-2}$ , and calculating the integral reveals the leading behavior  $\tilde{\theta} \propto D$  for  $D \gg 1$ . Since the square bracket becomes arbitrarily small, for  $z = r_x + i0$  and  $z = r_x - i0$ , in the “central” region  $1 \ll r_x \ll D$ , the quantity  $\delta\theta$  represents the contribution to the torque from the ends of the needles, and  $\delta\theta$  can be written as a sum of two expressions. One of these,  $\delta_{<}\theta$ , corresponds to semi-infinite needles extending from 1 to  $|\infty|$  and from  $e^{i\alpha}$  to  $e^{i\alpha}|\infty|$ . The other contribution,  $\delta_{>}\theta$ , corresponds to needles extending from 0 to  $D$  and from 0 to  $e^{i\alpha}D$ . In the case of  $\delta_{<}\theta$ , the difference  $< T(z) > - < T(z) >_{\text{wedge}}$  for  $r_x \gg 1$  is proportional to  $r_x^{-2-(2\pi/\alpha)}$ , if  $z = r_x + i0$ , and to  $r_x^{-2-(2\pi/(2\pi-\alpha))}$ , if  $z = r_x - i0$ . This follows from Eqs. (3.7)-(3.10) for the mirror-symmetric case considered here with  $b = 1$ . Thus, only  $r_x$  values

of order 1 contribute, and  $\delta_{<}\theta$  is proportional to  $D$  for  $D \rightarrow \infty$ . As for  $\delta_{>}\theta$ , its needle geometry can be mapped either by the dilatation  $z/D \rightarrow z$  to needles of length 1 or by the inversion  $D/z \rightarrow z$  to the needle geometry of  $\delta_{<}$ . Either way, one realizes that  $\delta_{>}\theta$  is of order 1. The plausible assumption that the  $D$  dependence from  $\tilde{\theta}$  and  $\delta_{<}\theta$  does not cancel leads to the predictions  $\theta \propto D$  and  $\Theta \propto D$  for  $D \rightarrow \infty$ , in agreement with the numerical results for case (C) in Sec. V.

For a needle  $I$  with ends at  $e^{i\Phi}$  and  $(D+1)e^{i\Phi}$  in the upper half plane, similar arguments also imply  $\Theta \propto D$  for  $D \rightarrow \infty$ . This is consistent with the numerical results for the torque in Fig. 9 for  $\Phi$  close to  $\pi/3$ .



## Appendix D: NEEDLES WITH MIXED BOUNDARY CONDITIONS

### 1. Half plane with inhomogeneous boundary conditions

We begin with a discussion of  $\langle T(w) \rangle_{u_1, u_2, \dots, u_N}$  in the upper half  $w$  plane with boundary conditions on the real axis that alternate between  $+$  and  $-$  at the  $N$  points  $u_1, u_2, \dots, u_N$ . If, for example, the boundary condition for  $-\infty < u < u_1$  is  $+$ , then it is  $-$  for  $u_1 < u < u_2$ ,  $+$  for  $u_2 < u < u_3$ , etc. The stress tensor for such mixed boundary conditions is of interest in its own right and is also the starting point for studying the Casimir interaction of needles with mixed boundary conditions.

For  $N = 0$ ,  $\langle T(w) \rangle$  vanishes, and for  $N = 1, 2$  [37]

$$\langle T(w) \rangle_{u_1} = \frac{\tilde{t}}{(w - u_1)^2}, \quad \langle T(w) \rangle_{u_1, u_2} = \tilde{t} \left( \frac{1}{w - u_1} - \frac{1}{w - u_2} \right)^2, \quad \tilde{t} = \tilde{t}_{+-} \equiv 1/2. \quad (\text{D1})$$

For  $N = 3$

$$\begin{aligned} \langle T(w) \rangle_{u_1, u_2, u_3} &= \frac{1}{(u_{12})^{-1} - (u_{13})^{-1} + (u_{23})^{-1}} \times \\ &\times \left\{ [12] - [13] + [23] + (12, 3) - (13, 2) + (23, 1) \right\}, \\ [ab] &\equiv \frac{\langle T(w) \rangle_{u_a, u_b}}{u_{ab}}, \quad (ab, c) \equiv \frac{1}{u_{ab}} \langle T(w) \rangle_{u_c}, \end{aligned} \quad (\text{D2})$$

and for  $N = 4$

$$\begin{aligned} \langle T(w) \rangle_{u_1, u_2, u_3, u_4} &= \frac{1}{(u_{12}u_{34})^{-1} - (u_{13}u_{24})^{-1} + (u_{23}u_{14})^{-1}} \times \\ &\times \left( [12]/u_{34} - [13]/u_{24} + [14]/u_{23} + [23]/u_{14} - [24]/u_{13} + [34]/u_{12} \right), \end{aligned} \quad (\text{D3})$$

respectively, where  $u_{ab} = u_a - u_b$ . For  $N$  an arbitrary even integer  $\geq 4$

$$\begin{aligned} \langle T(w) \rangle_{u_1, u_2, \dots, u_N} &= \left( \text{Pf}^{(N)} \frac{1}{u_{ij}} \right)^{-1} \times \frac{\partial}{\partial \lambda} \text{Pf}^{(N)} \left( \frac{1}{u_{ij}} + \lambda [ij] \right) \Big|_{\lambda=0} \\ &= \left( \text{Pf}^{(N)} \frac{1}{u_{ij}} \right)^{-1} \times \sum_{1 \leq a < b \leq N} (-1)^{a+b+1} [ab] \text{Pf}_{ab}^{(N \rightarrow N-2)} \frac{1}{u_{ij}}. \end{aligned} \quad (\text{D4})$$

Here  $\text{Pf}^{(N)} A_{ij}$  is the Pfaffian [55] of the  $N \times N$  antisymmetric matrix with elements  $A_{ij} = -A_{ji}$ , the sum in (D4) contains  $\frac{1}{2}N(N-1)$  terms, and  $\text{Pf}_{ab}^{(N \rightarrow N-2)} A_{ij}$  is the Pfaffian of the  $(N-2) \times (N-2)$  matrix obtained from the  $N \times N$  matrix by removing the  $a$ th and  $b$ th rows and columns. In the limit  $u_N \rightarrow \infty$  Eq. (D4) yields the stress tensor for an arbitrary odd

number  $N - 1$  of switches. Equation (D3) follows from Eq. (D4) for  $N = 4$  and Eq. (D2) from Eq. (D3) in the limit  $u_4 \rightarrow \infty$ . Since the operator  $T$  is even in the order parameter field  $\phi$ ,  $\langle T(w) \rangle_{u_1, u_2, \dots, u_N}$  is unchanged on exchanging  $+$  and  $-$  in the boundary conditions.

Equation (D4) follows from the result

$$\begin{aligned} \langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle_{u_1, u_2, \dots, u_N} &= \left( \text{Pf}^{(N)} \frac{1}{u_{ij}} \right)^{-1} \times \\ &\times \langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle \text{Pf}^{(N)} \left[ \frac{1}{u_{ij}} \frac{\langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle_{u_i, u_j}}{\langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle} \right]. \end{aligned} \quad (\text{D5})$$

of Burkhardt and Guim [56] for the two-point correlations of the order parameter in the presence of mixed boundary conditions. In this expression the angular brackets without subscripts denote thermal averages for a homogeneous  $+$  or  $-$  boundary condition. In the limit that  $w_1$  is much closer to  $w_2$  than to the boundary, Eq. (D5) must be consistent with the OPE

$$\begin{aligned} \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) &\rightarrow |w_{12}|^{-1/4} \left\{ 1 - \frac{1}{2} |w_{12}| \epsilon(w, \bar{w}) + \frac{1}{4} [w_{12}^2 T(w) + \bar{w}_{12}^2 \bar{T}(\bar{w})] + \mathcal{O}(|w_{12}|^3) \right\}, \\ w_{12} \equiv w_1 - w_2 \quad , \quad w &\equiv (w_1 + w_2)/2, \end{aligned} \quad (\text{D6})$$

see, e.g., Eq. (2.39) and Sec. IIIC in Ref. [44]. Substituting the expansion (D6) in all the averages in Eq. (D5) and comparing the coefficients of  $|w_{12}|^{-1/4} w_{12}^2$  on the right and left hand sides leads to Eq. (D4).

## 2. Interaction of semi-infinite needles with mixed boundary conditions

Under the mapping (3.8) of the upper half  $w$  plane onto the  $z$  plane with two embedded semi-infinite needles, the intervals  $-\infty < u < -b$ ,  $-b < u < 0$  and  $0 < u < 1$ ,  $1 < u < +\infty$ , which we denote by (i), (ii) and (iii), (iv), map onto the edges of the semi-infinite needles  $II$  and  $I$ , respectively. In the notation of the preceding Subsection, we consider the following distributions of surface universality classes  $+$  and  $-$  along the  $u$ -axis:

- (1)  $N = 0$
- (2)  $N = 2$ ,  $u_1 = -b$ ,  $u_2 = 1$
- (3)  $N = 3$ ,  $u_1 = -b$ ,  $u_2 = 0$ ,  $u_3 = 1$
- (4)  $N = 1$ ,  $u_1 = 0$
- (5)  $N = 1$ ,  $u_1 = 1$

In cases (1) and (4), which were considered in Sec. IIIB, the boundary conditions on the

two edges of each needle are the same. In cases (2), (3), and (5), on the other hand, one or both of the needles has a different boundary condition on each of its two edges. The stress tensor averages given above allow us to calculate the force between these needles.

We illustrate the approach in the particularly simple case of *collinear* semi-infinite needles generated by the mapping (3.8) with  $b = 1$  and  $\alpha = \pi$ . Needles  $I$  and  $II$  occupy the portions  $-\infty < x < -|z(1)| = -4\mathcal{B}$  and  $0 < x < +\infty$ , respectively, of the  $x$ -axis, and the four intervals of the boundary of the  $w$  plane map onto the upper and lower edges of needles  $I$  and  $II$  according to

$$(i) \rightarrow II_{\text{lower}}, (ii) \rightarrow II_{\text{upper}}, (iii) \rightarrow I_{\text{upper}}, (iv) \rightarrow I_{\text{lower}}. \quad (\text{D7})$$

Starting with  $+$  at  $u = -\infty$ , Eq. (D7) implies

$$\begin{pmatrix} I_{\text{upper}} & II_{\text{upper}} \\ I_{\text{lower}} & II_{\text{lower}} \end{pmatrix} = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} - & - \\ + & + \end{pmatrix}, \begin{pmatrix} + & - \\ - & + \end{pmatrix}, \begin{pmatrix} - & + \\ - & + \end{pmatrix}, \begin{pmatrix} + & + \\ - & + \end{pmatrix} \quad (\text{D8})$$

in cases (1)-(5), respectively. The force acting on needle  $I$  follows from Eqs. (2.3) and (2.5), the collinear needle mapping  $z(w)$ , and the averages  $\langle T(w) \rangle \dots$  in Eqs. (D1) and (D2). The component  $f_x$  is given by [38]

$$16|z(1)|f_x/(k_B T) = 1, 1 + 16\tilde{t} = 9, -7, 1 - 32\tilde{t} = -15, 1 - 8\tilde{t} = -3; \quad \tilde{t} \equiv 1/2 \quad (\text{D9})$$

in cases (1)-(5), respectively, and the component  $f_y$  vanishes in all the cases except (5), where  $|z(1)|f_y/(k_B T) = -2\tilde{t}/\pi \equiv -1/\pi$ .

It is remarkable that in case (2) of Eqs. (D8) and (D9), the attraction is 9 times stronger than in case (1)! To help understand this result, note that for the same *nonvanishing* distance  $|z(1)|$  between the needle tips, the free energy is greater in case (2) than in case (1), since in case (2) the spins change direction near the needle tips. However, when the tips touch, the free energy is the same in cases (1) and (2), since the upper and lower halves of the  $z$  plane are decoupled. Thus, the free energy varies more rapidly with the tip separation in case (2).

### 3. Semi-infinite needle perpendicular to an infinite needle

Next we consider a semi-infinite needle  $I$  in the upper half  $z$  plane oriented *perpendicular* to an infinite needle  $II$  on the  $x$  axis, as described by Eq. (3.2) with  $\alpha = \pi/2$ . The tip of

needle  $I$  is at  $z = z(1) = 4\mathcal{A}i$ , and the pre-image of the origin  $z = 0$  is at  $w = u = -1$ . Allowing for both a homogeneous boundary (+ for all  $x$ ) and a boundary with a “chemical step” at the origin (i.e., a mixed boundary with + for  $x < 0$  and  $-$  for  $x > 0$ ), and allowing for different boundary conditions on the right and left edges of needle  $I$ , we consider the six cases

$$\begin{pmatrix} I_{\text{left}} & I_{\text{right}} \\ II_{\text{left}} & II_{\text{right}} \end{pmatrix} = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \begin{pmatrix} - & + \\ + & - \end{pmatrix}, \begin{pmatrix} - & - \\ + & + \end{pmatrix}, \begin{pmatrix} - & + \\ + & + \end{pmatrix}, \begin{pmatrix} + & + \\ + & - \end{pmatrix} \quad (\text{D10})$$

The stress tensor averages  $\langle T(w) \rangle$  in the first five cases are the five defined in the first paragraph of Subsec. D 2 with  $b = 1$  and in the sixth case  $N = 2, u_1 = -b = -1, u_2 = 0$ , corresponding to

$$(i) \rightarrow II_{\text{left}}, (ii) \rightarrow II_{\text{right}}, (iii) \rightarrow I_{\text{right}}, (iv) \rightarrow I_{\text{left}}. \quad (\text{D11})$$

Together with Eqs. (3.1), (2.3), and (2.5) this leads to [38]

$$\begin{aligned} 32|z(1)|f_y/(k_B T) &= -3, -3(1 + 16\tilde{t}) = -27, 37, -3 + 128\tilde{t} = 61, \\ 3(-1 + 16\tilde{t}) &= 21, -3 + 32\tilde{t} = 13; \quad \tilde{t} \equiv 1/2. \end{aligned} \quad (\text{D12})$$

The parallel force component  $f_x$  vanishes in cases (1)-(5), and in case (6),  $|z(1)|f_x/(k_B T) = -2\tilde{t} \equiv -1$ . The factor 9 increase in attraction on going from (1) to (2) has an explanation similar to the one below Eq. (D9).

In principle, one can calculate the force for arbitrary configurations of two semi-infinite or infinite needles with an arbitrary configuration of “chemical steps” with this approach.

As a final example, we consider the Casimir force exerted on the semi-infinite needle  $I$  by the boundary  $II$  of the upper half  $z$  plane in the presence of chemical steps at two arbitrary points

$$x_1 \equiv X_1|z(1)| < x_2 \equiv X_2|z(1)|, \quad (\text{D13})$$

which separate the  $x$  axis into regions with  $+, -, +$  boundary conditions. Needle  $I$  has boundary condition  $+$  on both of its edges and extends along the  $y$  axis from  $y = |z(1)|$  to  $y = +\infty$ . The arrangement is reminiscent of an atomic force microscope probing an inhomogeneous boundary. The force follows from the mapping (3.2) with  $\alpha = \pi/2$  and the stress tensor in (D1) with  $N = 2$  and  $u_1 = -|u_1|, u_2 = -|u_2|$ , where

$$|u_j| = 1 + 2X_j^2 - 2X_j\sqrt{1 + X_j^2}; \quad j = 1, 2, \quad (\text{D14})$$

and the calculation yields.

$$|z(1)|\frac{f_x}{k_B T} = 4\tilde{t}\left\{-\frac{2}{|u_1| - |u_2|}\left[\frac{|u_1|^{3/2}}{1 + |u_1|} - \frac{|u_2|^{3/2}}{1 + |u_2|}\right] + \sum_{j=1}^2 \frac{|u_j|^{1/2}(|u_j| + 3)}{2(1 + |u_j|)^2}\right\},$$

$$|z(1)|\frac{f_y}{k_B T} = -\frac{3}{32} + 4\tilde{t}\left(\frac{1}{1 + |u_1|} - \frac{1}{1 + |u_2|}\right)^2, \quad (\text{D15})$$

where  $|u_j|$  is defined in Eq. (D14).

To get a feeling for the result, we discuss two special cases.

(A) Boundary with a single step: In the limit  $x_2 \rightarrow +\infty$ , i.e.  $|u_2| \rightarrow 0$ , only the single step on the boundary at  $x_1$  remains. It separates regions with  $+$  and  $-$  boundary conditions to its left and right, respectively. The corresponding force on needle  $I$  is

$$|z(1)|\frac{f_x}{k_B T} = -2\tilde{t}\frac{|u_1|^{1/2}(3|u_1| + 1)}{(1 + |u_1|)^2},$$

$$|z(1)|\frac{f_y}{k_B T} = -\frac{3}{32} + 4\tilde{t}\frac{|u_1|^2}{(1 + |u_1|)^2}, \quad (\text{D16})$$

with  $|u_1|$  given by Eq. (D14). While the parallel force component  $f_x$  is negative for all  $x_1$ , the perpendicular component  $f_y$  changes sign from positive to negative on increasing  $x_1$  beyond a critical value of the order of  $|z(1)|$ . This is expected, since needle  $I$  with its  $+$  edges is attracted to the  $+$  region and repelled by the  $-$  region of the boundary. For  $x_1 \rightarrow [-\infty, 0, +\infty]$ ,  $|z(1)|f_x/(k_B T) \rightarrow -\tilde{t}[3/|X_1|, 2, 1/X_1]$ , and  $|z(1)|f_y/(k_B T) \rightarrow -(3/32) + \tilde{t}[4 - (2/X_1^2), 1, 1/(4X_1^4)]$ . For  $x_1 = 0$  one recovers case (6) defined below Eq. (D10), and Eq. (D16) reproduces the corresponding force components given in the paragraph containing Eq. (D12). For  $x_1 \rightarrow -\infty$  and  $x_1 \rightarrow +\infty$ , Eq. (D16) approaches the force in cases (4) and (1) of Eq. (D12)

(B) Boundary with two steps at equal distances from the needle: Since the configuration, with steps at  $\pm x_1$  separating the  $x$  axis into regions  $+, -, +$  is mirror symmetric about the  $y$ -axis, the parallel component  $f_x$  of the force vanishes. The perpendicular component follows from Eqs. (D14) and (D15), which yield  $|u_2| = 1/|u_1|$  and

$$|z(1)|\frac{f_y}{k_B T} = -\frac{3}{32} + 4\tilde{t}\frac{X_1^2}{1 + X_1^2}. \quad (\text{D17})$$

For  $x_1 = 0$  the boundary with two steps reduces to a homogeneous  $+$  boundary, and we are back to case (1) of Eqs. (D10) and (D12). For a large distance between the steps,  $|x_1| \gg |z(1)|$ , Eq. (D17) yields

$$|z(1)|\frac{f_y}{k_B T} \rightarrow -\frac{3}{32} + 4\tilde{t} - \frac{4\tilde{t}}{X_1^2}. \quad (\text{D18})$$

Here the first two terms on the right hand side represent the force exerted on the needle by a homogeneous  $-$  boundary, and the third term is contributed by the  $+$  boundaries beyond the two distant steps. As expected, the latter contribution is twice the corresponding contribution  $-2\tilde{t}/X_1^2$  of a single distant step, given below Eq. (D16).

For switches of the boundary universality class between  $+$  and  $O$  instead of  $+$  and  $-$ ,  $\langle T(w) \rangle_{u_1}$  and  $\langle T(w) \rangle_{u_1, u_2}$  are again given by Eq. (D1), [37], but with  $\tilde{t} = \tilde{t}_{+O} = 1/16$  instead of  $\tilde{t} = \tilde{t}_{+-} = 1/2$ . Thus, all of the results of this Appendix which are based on the stress tensor for  $N = 1$  or  $N = 2$  hold, with the appropriate value of  $\tilde{t}$ , for  $+O$  as well as  $+-$  switches in the boundary conditions.

- 
- [1] C. Hertlein, L. Helden, A. Gambassi, S. Dietrich, and C. Bechinger, *Nature* **451**, 172 (2008).
  - [2] A. Gambassi, A. Maciolek, C. Hertlein, U. Nellen, L. Helden, C. Bechinger, and S. Dietrich, *Phys. Rev. E* **80**, 061143 (2009).
  - [3] M. E. Fisher and P. G. de Gennes, *C. R. Acad. Sci. Paris*, **287**, B-207 (1978).
  - [4] M. Krech, *The Casimir Effect in Critical Systems* (World Scientific, Singapore, 1994).
  - [5] I. Brankov, D. M. Danchev, and N. S. Tonchev, *Theory of Critical Phenomena in Finite-Size Systems: Scaling and Quantum Effects* (World Scientific, Singapore, 2000).
  - [6] A. Gambassi, *J. Phys.: Conf. Ser.* **161**, 012037 (2009).
  - [7] M. Kardar and R. Golestanian, *Rev. Mod. Phys.* **71**, 1233 (1999).
  - [8] M. Bordag, U. Mohideen, and V. M. Mostepanenko, *Phys. Rep.* **353**, 1205 (2001).
  - [9] T. Emig, N. Graham, R.L. Jaffe, and M. Kardar, *Phys. Rev. A* **79**, 054901 (2009).
  - [10] H. Gies and K. Klingmueller, *Phys. Rev. Lett.* **97**, 220405 (2006); A. Weber and H. Gies, *Phys. Rev. D* **80**, 065033 (2009).
  - [11] E. Eisenriegler, in *Soft Matter*, edited by G. Gompper and M. Schick (Wiley-VCH, Weinheim, 2006), Vol. **2**, p.87.
  - [12] E. Eisenriegler, *J. Chem. Phys.* **124**, 144912 (2006); *J. Chem. Phys.* **125**, 204903 (2006).
  - [13] F. Schlesener, A. Hanke, R. Klimpel, and S. Dietrich, *Phys. Rev. E* **63**, 041803 (2001).
  - [14] D. Kleshchanok, R. Tuinier, P. Lang, *J. Phys.: Cond. Mat.* **20**, 073101 (2008).
  - [15] M. F. Maghrebi, Y. Kantor, and M. Kardar, *EPL* **96**, 66002 (2011); *Phys. Rev. E* **86**, 061801 (2012).
  - [16] P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University, Ithaca, 1979).
  - [17] R. Garcia and M. H. W. Chan, *Phys. Rev. Lett.* **83**, 1187 (1999).
  - [18] A. Ganshin, S. Scheidemantel, R. Garcia, and M. H. W. Chan, *Phys. Rev. Lett.* **97**, 075301 (2006).
  - [19] O. Vasilyev, A. Gambassi, A. Maciolek, and S. Dietrich, *EPL* **80**, 60009 (2007).
  - [20] U. Nellen, L. Helden, and C. Bechinger, *EPL* **88**, 26001 (2009).
  - [21] (a) K. Binder, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, London, 1983), Vol. **8**, p. 1; (b) H. W. Diehl, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, London, 1986),

- Vol. **10**, p. 76; (c) H. W. Diehl, Int. J. Mod. Phys. B **11**, 3503 (1997).
- [22] M. Hasenbusch, Phys. Rev. B **81**, 165412 (2010).
  - [23] R. Evans and J. Stecki, Phys. Rev. B **49**, 8842 (1994).
  - [24] T. W. Burkhardt and E. Eisenriegler, Phys. Rev. Lett. **74**, 3189 (1995); **78**, 2867 (1997); E. Eisenriegler and U. Rietschel, Phys. Rev. B **51**, 13717 (1995).
  - [25] S. Kondrat, L. Harnau, and S. Dietrich, J. Chem. Phys. **131**, 204902 (2009).
  - [26] T. Baumgart, A. T. Hammond, P. Sengupta, S. T. Hess, D. A. Holowka, B. A. Baird, and W. W. Webb, Proc. Natl. Acad. Sci. USA **104**, 3156 (2007); S. L. Veatch, O. Soubias, S. L. Keller, and K. Gawrisch, Proc. Natl. Acad. Sci. USA **104**, 17650 (2007).
  - [27] B. B. Machta, S. L. Veatch, and J. P. Sethna, Phys. Rev. Lett. **109**, 138101 (2012).
  - [28] J. L. Cardy, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1986), Vol. 11, p. 55; J. L. Cardy, Nucl. Phys. B **240**, 514 (1984).
  - [29] O. A. Vasilyev, E. Eisenriegler, and S. Dietrich, Phys. Rev. E **88**, 012137 (2013).
  - [30] G. Bimonte, T. Emig, and M. Kardar, EPL **104**, 21001 (2013).
  - [31] J. L. Cardy, Nucl. Phys. B **275** FS17, 200 (1986).
  - [32] Needle shaped particles have also been considered in the context of quantum spin chains. See J. M. Stephan, J. Stat. Mech. P05010 (2014); M. A. Rajabpour, arXiv: 1512.01052v1; and references therein.
  - [33] We use the terms “distant”, “small”, and “short” interchangeably to describe a needle that is short compared with its distance from the rest of the system but large on a microscopic scale.
  - [34] K.G Wilson and J.B Kogut, Phys. Rep. C **12**, 75 (1974); A. Z. Patashinskii and V. L. Pokrovskii, *Fluctuation Theory of Phase Transitions* (International Series in Natural Philosophy, Vol. **98**, Pergamon, New York, 1979); A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nuclear Phys. B **241**, 333 (1984).
  - [35] The free energy of interaction  $\delta F$  is the free energy required to move the two needles from infinite separation to the configuration of interest.
  - [36] H. Kober, *Dictionary of Conformal Representations*, Dover 1957. See §12.1, 12.3, 13.11, and 13.12.
  - [37] T. W. Burkhardt and T. Xue, Nucl. Phys. B **345**, 653 (1991).
  - [38] A useful formula for evaluating the integrals in Sec. III and Appendix D is

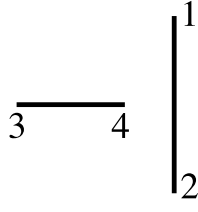


$\int_0^\infty dU U^{\mu-1} [(U+b)(U-a+i0)]^{-1} = -\pi \csc(\pi\mu) [b^{\mu-1} + i^{2\mu} a^{\mu-1}] / (a+b)$ , where  $a$  and  $b$  are positive.

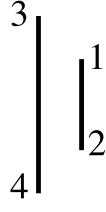
- [39] N. I. Akhiezer, Translations of Mathematical Monographs **79**, *Elements of the theory of elliptic functions* (American Mathematical Society, 1970). We follow Akhiezer's definition of the function  $\vartheta_1(v)$ , which is related by  $\vartheta_1(v) = \vartheta_1^{\text{alternate}}(\pi v)$  to the alternate function of Abramowitz and Stegun, Gradshteyn and Ryzhik, and *Mathematica*.
- [40] For finite needles that nearly touch,  $h$  is very close to 1. This regime can be studied analytically with the help of "Jacobi's imaginary transformation," which relates  $\vartheta_1$  with parameter  $h$  to  $\vartheta_1$  with parameter  $h' \equiv \exp(-\pi^2/\ln(1/h))$ , since  $h' \rightarrow 0$  for  $h \rightarrow 1$ . For details see §22 of Akhiezer's book [39].
- [41] Rotating configuration (D) of Sec. IV A 1 counterclockwise by an angle of  $\pi - \Phi_{12}$ , where  $0 < \Phi_{12} < \pi$ , yields a configuration (D') with needle vectors  $z_{12} = -|z_{12}|$  and  $z_{34} = |z_{34}|$  antiparallel and parallel to the real axis, respectively, and with needle  $I$  located above needle  $II$ , as shown in Fig. 1-(D) and 1-(D'). In Sec. V and Appendix C this new orientation is more convenient. The angle  $\Phi_{12}$  is related to the new  $x$  and  $y$  components by  $\cot \Phi_{12} = -(r_{1,x} - r_{4,x})/W$ ,  $W \equiv r_{1,y} - r_{4,y} = |r_{1,y} - r_{4,y}|$ .
- [42] At bulk criticality the free energy of an annulus bounded by two concentric circles with radii  $h\rho$  and  $\rho$  contains a scale-free contribution  $f_{\text{annulus}}(h)$  which is independent of  $\rho$ , depends only on  $h$ , and vanishes for  $h \rightarrow 0$ . Since  $h(d/dh)f_{\text{annulus}}/(k_B T) = -t(h)$ , this contribution is directly related to  $t(h)$  in Eqs. (4.22)-(4.24). While  $t(h \ll 1)$  is given in Eqs. (4.26) and (4.27) and is relevant for the small particle expansion,  $t(h \rightarrow 1) = -2\pi(1-h)^{-2}\Delta$  is determined by the interaction between the parallel boundaries of an infinitely long strip, where the Derjaguin approximation is exact [24]. Here  $\Delta = \pi[-1, -1, 23, 2]/48$  is the amplitude [28] in the Casimir contribution  $\Delta/W$  to the free energy per  $k_B T$  and per unit length of the strip with width  $W$  and boundary universality classes  $OO, ++, +-, O+$  and is related [37] via  $\Delta = \pi(\tilde{t} - 1/48)$  to the amplitude  $\tilde{t}$  in Eq. (3.4). This is consistent with  $h \rightarrow 1$  for needles much longer than their minimum separation.
- [43] That  $\langle T(w) \rangle$  in (4.22) does not change on swapping the surface universality classes of the two boundary circles follows from its transformation under the inversion  $w \rightarrow h/w$ . This result and Eq. (2.1) imply that  $\langle T(z) \rangle$  for a given configuration of two needles is also unaffected by an exchange of their surface universality classes.

- [44] E. Eisenriegler, J. Chem. Phys. **121**, 3299 (2004).
- [45] Only in exceptional cases does the force between two finite needles have a simple analytic form, free of special functions, for arbitrary distance to size ratios. One such exception is the collinear configuration (i) of Sec. IV A 1, where the force on needle  $I$  is given by  $f_x/(k_B T) = \pm\{8c[(c/D) + 1][(c/D) + 2]\}^{-1}$  for  $O+$  (upper sign) and  $OO$  (lower sign) boundaries. Here  $c = z_2 - z_4 \equiv r_{2,x} - r_{4,x}$  is the distance between the closest points of the needles. This result follows from stress tensor averages  $\langle T(z) \rangle$  given in Appendix B of Ref. [29] on using our Eqs. (A8) and (A11).
- [46] Since translating or rotating particle  $I$  relative to particle  $II$  as described in (A3)-(A5) does not alter the nonuniversal contributions to the bulk and surface free energies and to the free energy of needle ends,  $F_{\tilde{G}} - F_G$  can be identified with the change in the total free energy.
- [47] We use the same normalization of the Cartesian stress tensor  $T_{kl}$  in Eqs. (A2) and (B18) as in field theories in arbitrary spatial dimensions, see e.g. L. S. Brown, Ann. Phys. (NY) **126**, 135 (1970). In two dimensions this leads to  $T(r_x + ir_y) = -\pi[T_{xx}(r_x, r_y) - iT_{xy}(r_x, r_y)]$  and  $\bar{T}(r_x - ir_y) = -\pi[T_{xx}(r_x, r_y) + iT_{xy}(r_x, r_y)]$ , where  $T$  and  $\bar{T}$  are the components of the complex stress tensor of Ref. [28]. With this normalization,  $\langle T(z)T(z') \rangle_{\text{bulk}} = (1/4)(z - z')^{-4}$ ,  $\langle T\bar{T} \rangle_{\text{bulk}} = 0$ , and  $\langle \bar{T}T \rangle_{\text{bulk}}$  is the complex conjugate of  $\langle TT \rangle_{\text{bulk}}$ . Cardy's Cartesian tensor  $T_{kl}^{(\text{Cardy})} = -2\pi T_{kl}$  in two dimensions differs from ours by a factor  $-2\pi$ .
- [48] The  $\tilde{T}(J)$ -term in Eq. (B16) is independent of  $H_J$  since at large distances it must reproduce the stress-tensor profile  $\langle T(z) \rangle_{\text{single needle}}$  around the *single* needle  $J$ , and this single-needle profile is independent of  $H_J$ . This is because  $\langle T(z) \rangle_{\text{single needle}}$  follows from a conformal mapping  $z(w)$  of the stress tensor  $\langle T(w) \rangle_{\text{half plane}}$  for the half plane with a uniform boundary condition, which vanishes [28], independent of  $H_J$ . Thus,  $\langle T(z) \rangle_{\text{single needle}}$  is entirely determined by the purely geometrical  $S(w)$  term in the transformation (2.1).
- [49] Due to a possible term  $\propto \partial^4 \phi$  in the ellipsis of Eq. (B16), for needle classes  $++$  and  $+-$  we expect a term of order (needle length  $/|\mathbf{r}_I - \mathbf{r}_{II}|$ ) raised to the power  $17/4$  in the ellipsis of Eq. (B25). Since this power is quite close to the power 4 coming from  $\mathcal{E}$  and  $\mathcal{H}$ , we have not shown the explicit form of the  $\mathcal{O}(B^{-4})$  term in Eqs. (B33) and (B36).
- [50] The special case of *symmetric* parallel semi-infinite needles, corresponding to Eq. (C6) with  $b = 1$ , is considered in §12.3 of Kober's book [36].
- [51] The expression for  $f_x$  for parallel semi-infinite needles derived in Appendix C also follows from

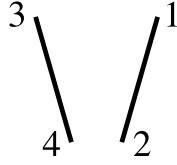
- the results of Sec. IIIB for the force between needles which form a nonvanishing angle  $\alpha > 0$ . Before taking the limit  $\alpha \searrow 0$ , one replaces the positive real constant  $\mathcal{B}$  in Eqs. (3.7)-(3.11) by  $|\mathcal{B}|e^{-i\alpha/2}$ , so that needles *I* and *II* form angles  $\alpha/2$  and  $-\alpha/2$  with the  $x$  axis and from the outset  $f_x$  is an odd function of the difference of the  $x$  coordinates of the needle tips.
- [52] See F. P. Toldin, M. Troendle, and S. Dietrich, J. Phys. Cond. Mat. **27**, 214010 (2015) and references therein.
  - [53] F. Soyka, O. Zvyagolskaya, C. Hertlein, L. Helden, and C. Bechinger, Phys. Rev. Lett. **101**, 208301 (2008).
  - [54] C. Yolcu, I. Z. Rothstein, and M. Deserno, EPL **96**, 20003 (2011); R. C. Haussman and M. Deserno, Phys. Rev. **89**, 062102 (2014).
  - [55] B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, MA, 1973).
  - [56] T. W. Burkhardt and I. Guim, Phys. Rev. B **47**, 14306 (1993). See in particular Eq. (17).



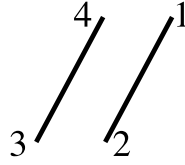
(A)



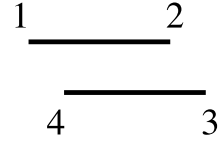
(B)



(C)



(D)



(D')

FIG. 1: Simple configurations of two needles defined in Sec. IV A 1 for which the six mapping parameters  $C, h, \varphi_1, \dots, \varphi_4$  of the conformal mapping are restricted to subspaces of lower dimension. Configuration (D') is the same as (D), apart from a rotation to orient the needles along the  $x$  axis. Our results for the force and torque in configurations (A), (B), (C), and (D') are presented in Figs. 2-6.

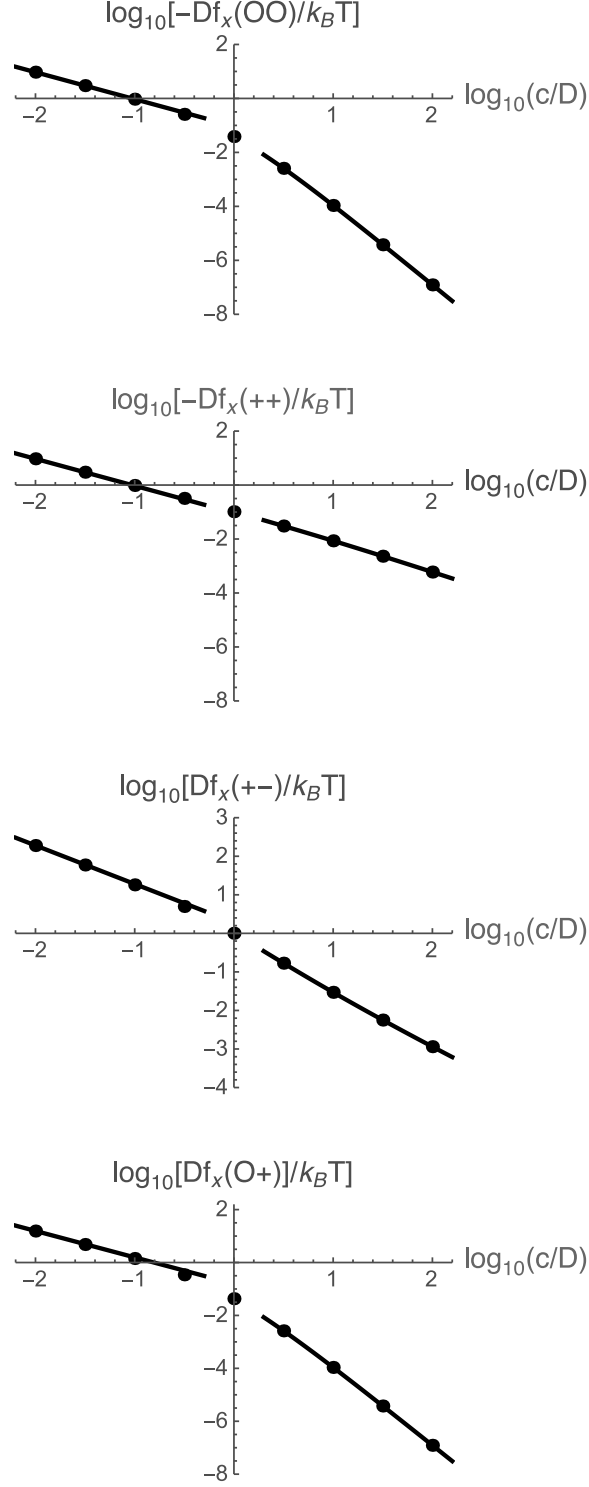


FIG. 2: Component  $f_x$  of the force exerted on needle  $I$  by needle  $II$  for needles of equal length  $D$  in the symmetric-perpendicular configuration (A) shown in Fig. 1. Here  $c = z_I - z_4$  is the distance from the right tip of needle  $II$  to the midpoint of needle  $I$ . The points indicate exact numerical results, and the two curves show the asymptotic form for large and small  $c/D$ . The force component  $f_y$  and the torque vanish due to symmetry. For more details see Sec. V.

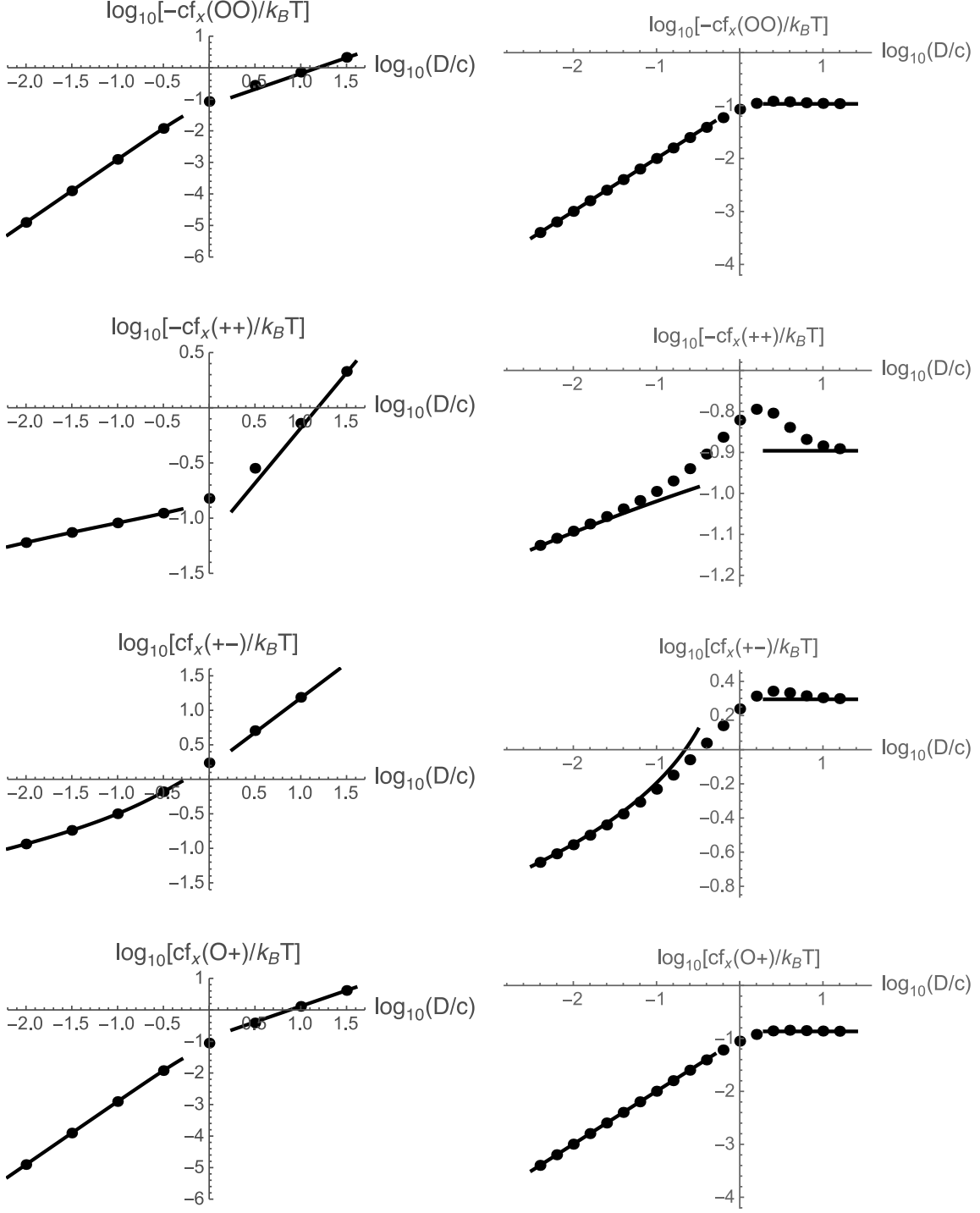


FIG. 3: Component  $f_x$  of the force exerted on needle  $I$  by needle  $II$  for needles with separation  $c = z_I - z_{II}$  in the symmetric-parallel configuration (B) in Fig. 1. The results in the left and right columns are for needles of the same length  $D$  and for needles with different lengths,  $c$  and  $D$ , respectively. The points indicate exact numerical results, and the curves show the asymptotic form for large and small  $D/c$ . The force component  $f_y$  and the torque vanish due to symmetry. For more details see Sec. V.

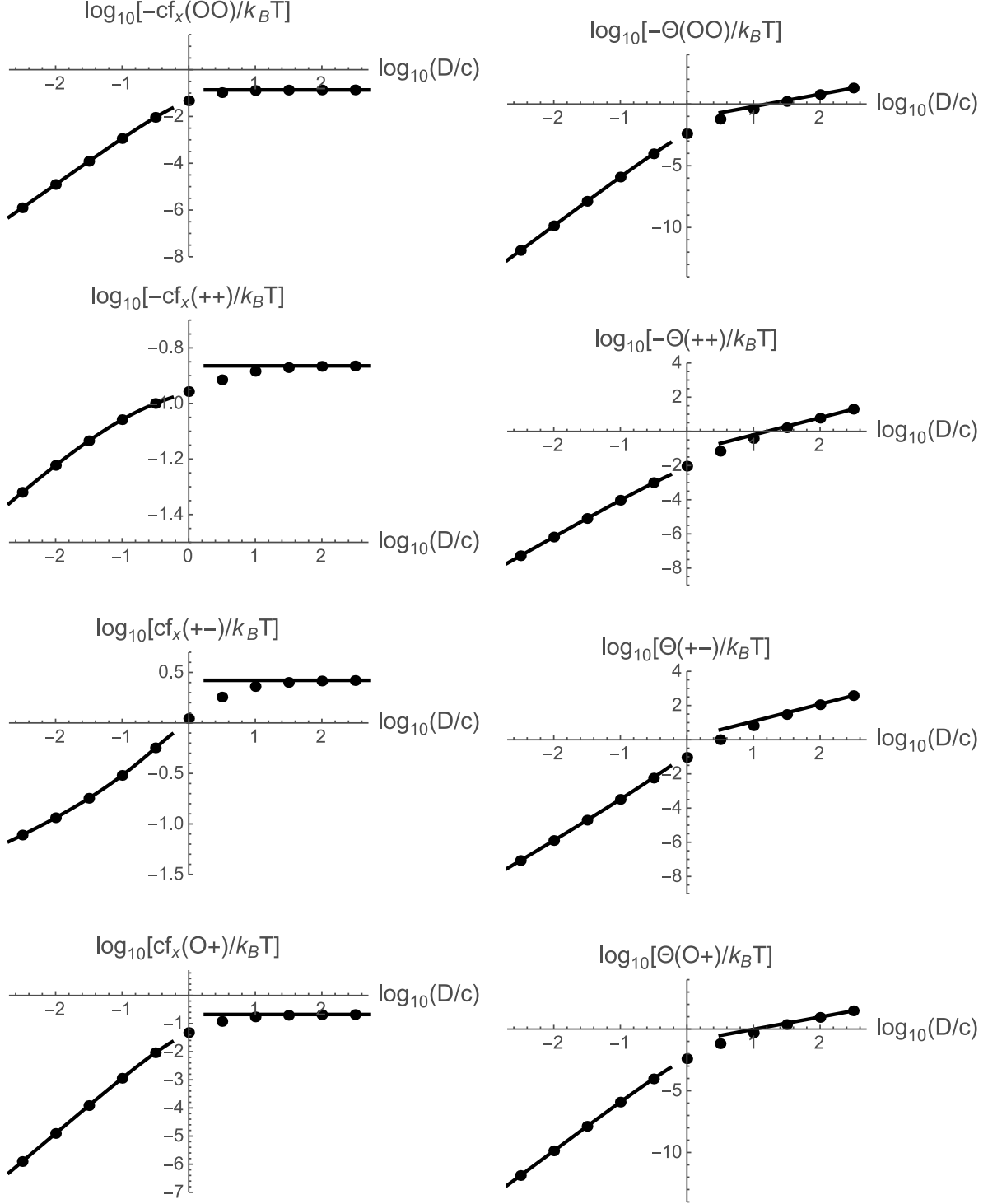


FIG. 4: Force  $f_x$  and torque  $\Theta$  exerted on needle  $I$  by needle  $II$  for needles of equal length  $D$  in the mirror-symmetric configuration (C) shown in Fig. 1. Here  $c = z_2 - z_4$  is the distance between the closest points of the needles, and the angle between them is  $\pi/5$ . The points indicate exact numerical results, and the two curves show the asymptotic form for large and small  $D/c$ . The force component  $f_y$  vanishes due to symmetry. For more details see Sec. V.

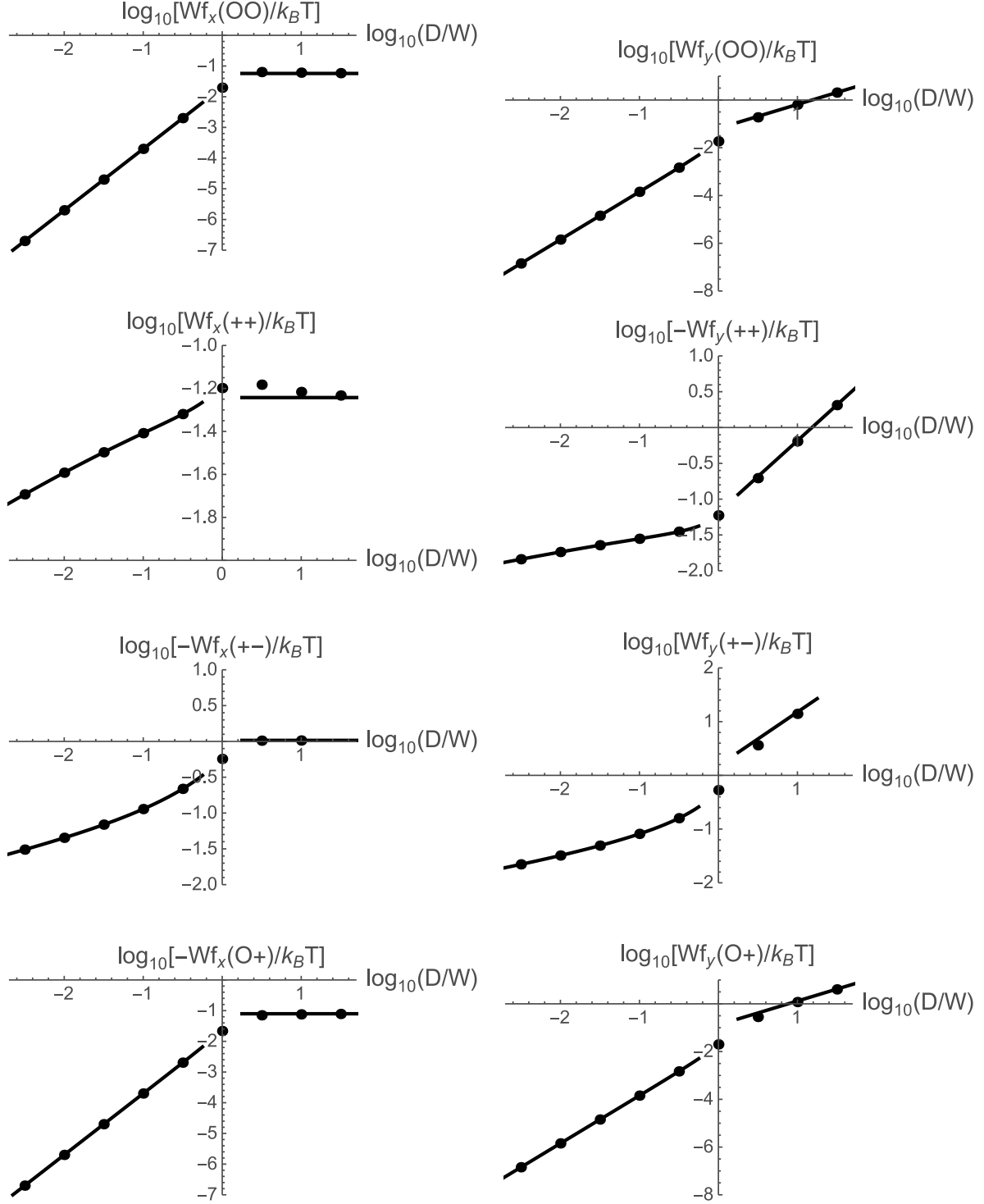


FIG. 5: Components  $f_x$ ,  $f_y$  of the force exerted on needle  $I$  by needle  $II$  for needles of equal length  $D$  in the antiparallel configuration ( $D'$ ) shown in Fig. 1. Here  $W = r_{1,y} - r_{4,y}$  is the vertical separation of the needles, and results are shown for the fixed ratio  $(r_{4,x} - r_{1,x})/W = 1.4$ . The points indicate exact numerical results, and the two curves show the asymptotic form for large and small  $D/W$ . For more details see Sec. V.



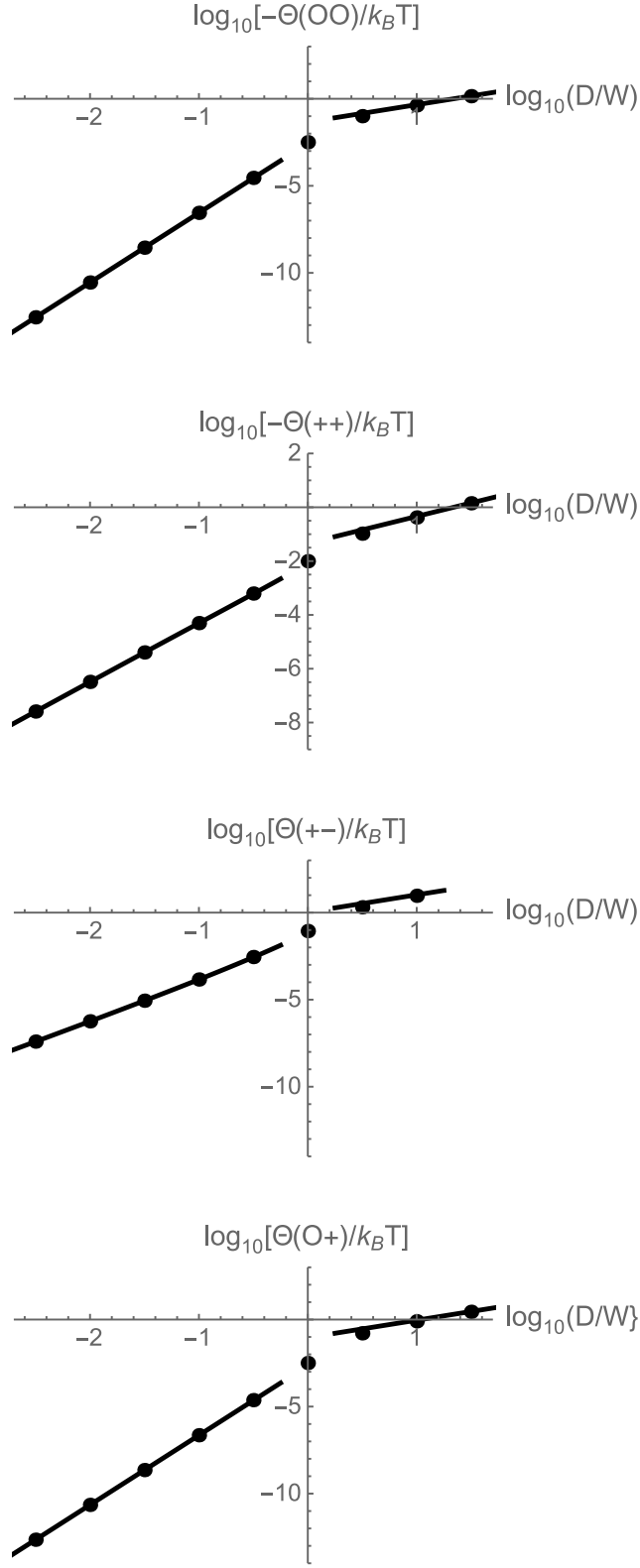


FIG. 6: Torque  $\Theta$  on needle  $I$  for the same antiparallel configuration considered in Fig. 5.

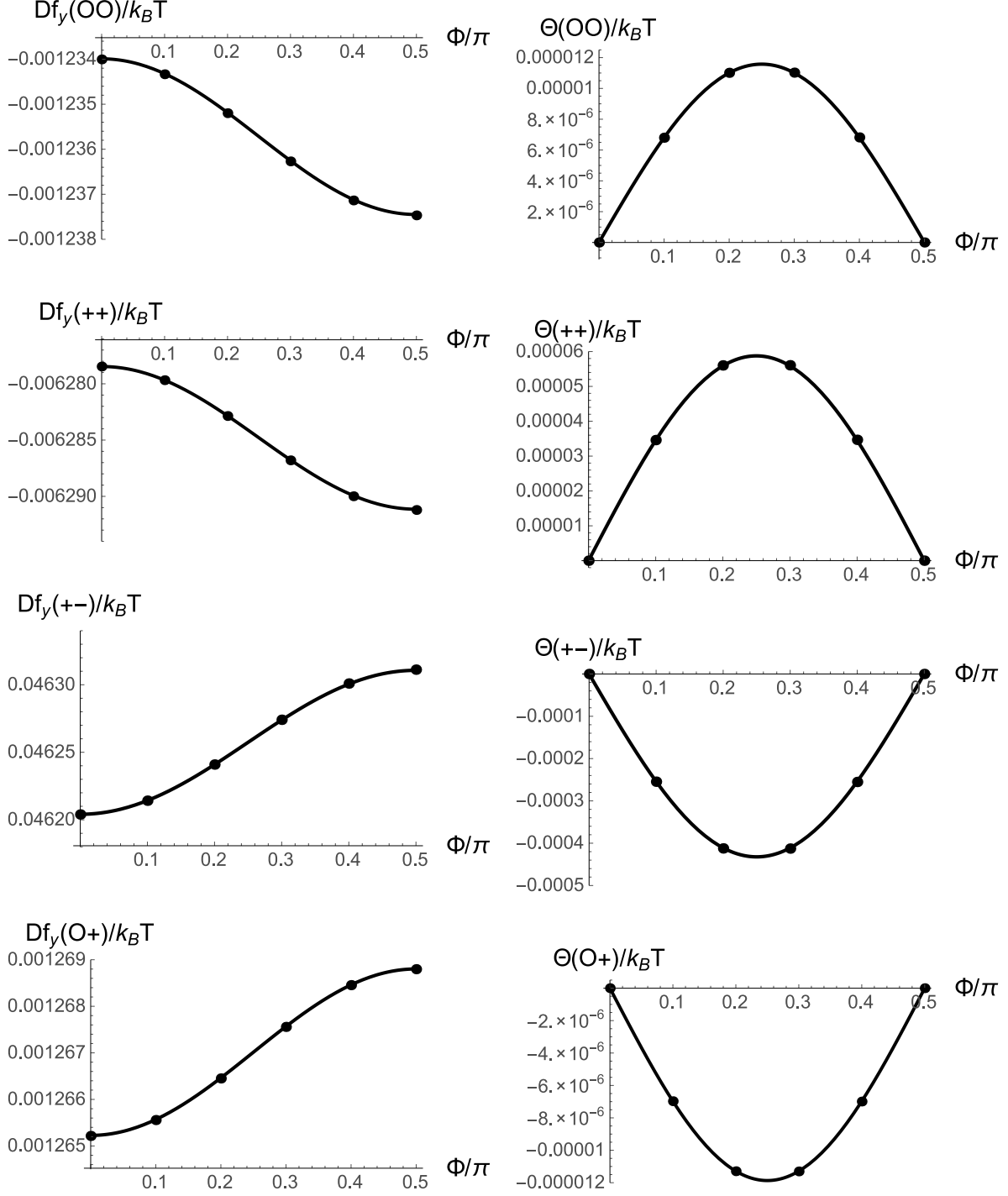


FIG. 7: Dependence of the force  $f_y$  and torque  $\Theta$  on a needle of length  $D$  in the upper half plane on the angle  $\Phi$  between the needle and the boundary. The reduced distance  $\tilde{B} = r_{I,y}/D$  of the needle midpoint from the boundary has the value  $\tilde{B} = 10$ . The points indicate exact numerical results, and the curves show the asymptotic predictions (B55) and (B56) of the operator expansion for a distant needle, i.e. for large  $\tilde{B}$ . The force component  $f_x$  vanishes. For more details see Sec. V.

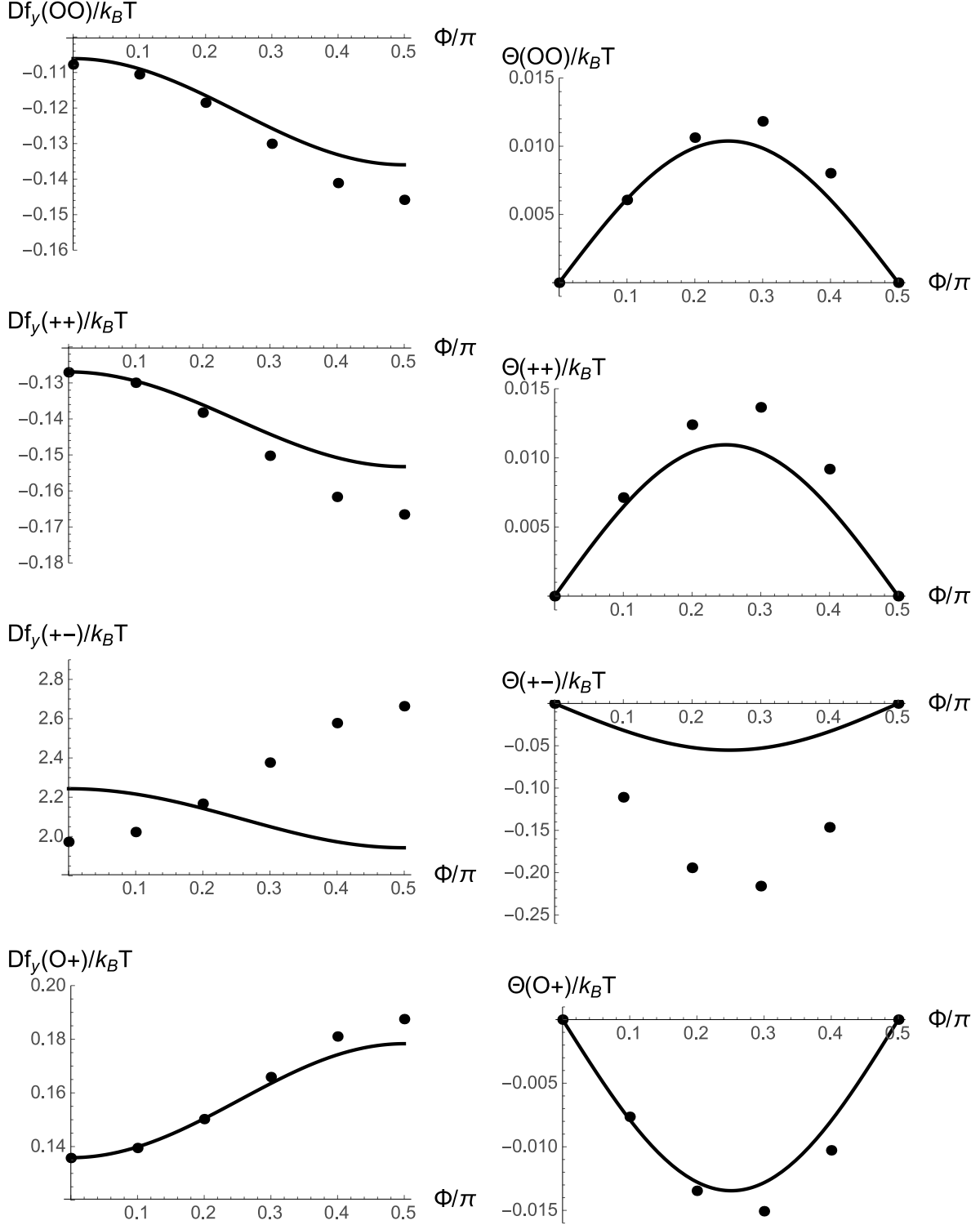


FIG. 8: Same as Fig. 7 except that the reduced distance has the value  $\tilde{B} = 1$  instead of 10. Since the distant needle condition  $\tilde{B} \gg 1$  is not satisfied, the exact numerical results (points) deviate significantly from the predictions (curves) of the operator expansion for a distant needle. For more details see Sec. V.

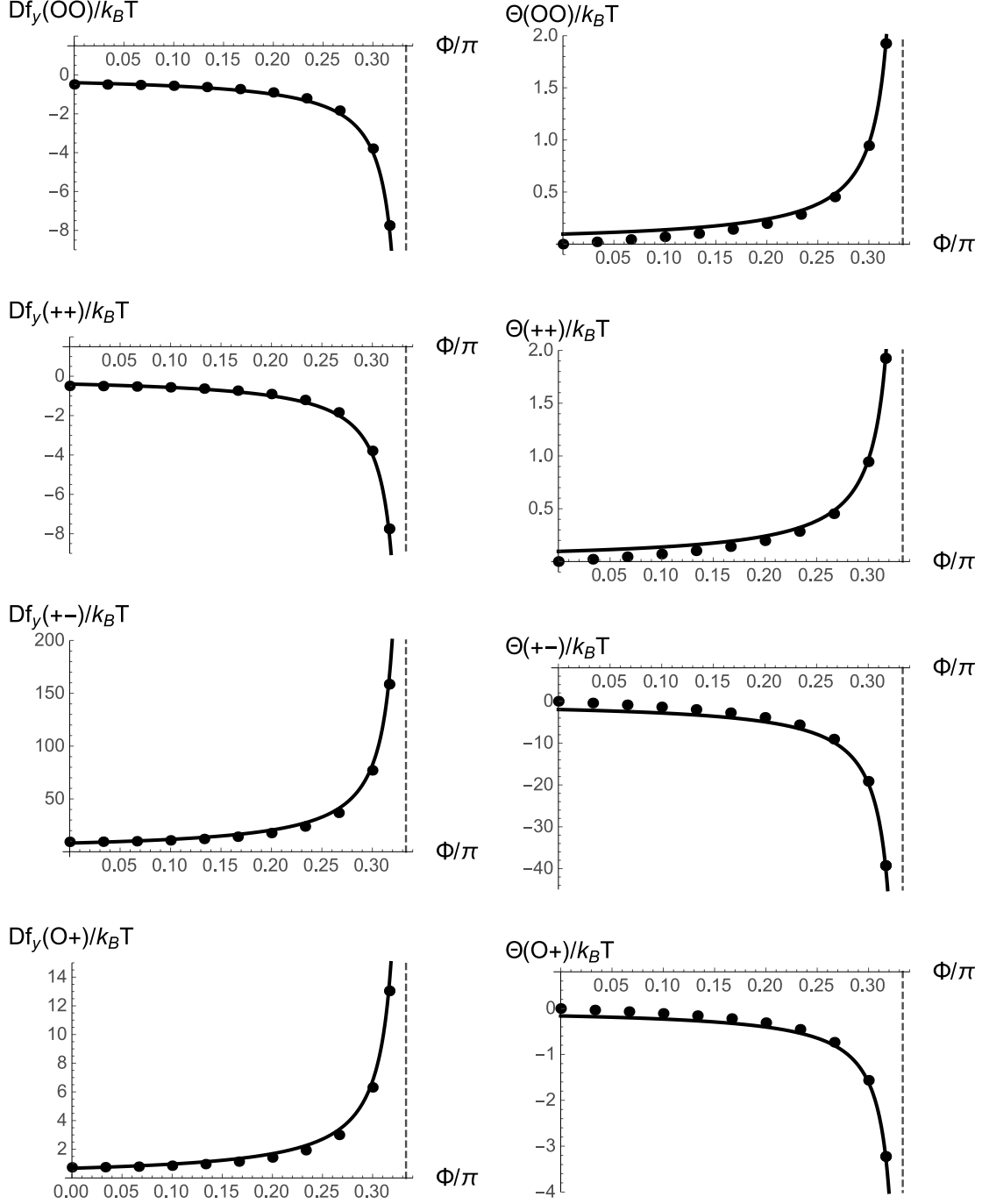


FIG. 9: Same as Fig. 7 except that  $\tilde{B} = \sqrt{3}/4 = 0.433$ . Both  $f_y$  and  $\Theta$  diverge as  $\Phi$  approaches  $\pi/3$ , the angle at which the needle tip touches the boundary. For  $\Phi$  close to  $\pi/3$ , the exact numerical results (points) for  $f_y$  agree with the asymptotic expression (curves) given in the last paragraph of Sec. V. Like  $f_y$ ,  $\Theta$  also appears to diverge as  $(\frac{\pi}{3} - \Phi)^{-1}$ . The exact numerical results for  $\Theta$  are compared with fits of the form  $\Theta_{\text{fit}}(\Phi) = A(\frac{\pi}{3} - \Phi)^{-1}$ , with  $A$  chosen to reproduce the rightmost point in each graph.